Chapter 2

2.1 (a) Write $\nabla(\nabla.\boldsymbol{v})$ in summation subscript notation. (b) Consider the viscous force per unit mass $\mathbf{e}$ introduced in equation 2.5.1. Write down the complete form of this by substituting equation 2.3.11, and then simplify for incompressible flow. (c) Determine $\partial x_i x_j/\partial x_i$ (*Hint: be careful*).

2.2 Write a short program to determine the maximum thickness and chordlength of a symmetric Joukowski airfoil as a function of the position of the center of the mapping circle to the left of the origin, i.e. $-Re\{\xi_1\}/C$ (see Figure 2.9A). Compare with the expressions in equation 2.3.37. Plot the true thickness to chord ratio as a function of $-Re\{\xi_1\}/C$.

2.3 Determine expressions for the velocity potential and streamfunction for the flow of a vertically upward free stream of unit velocity past a horizontal flat plate extending from $(-a, 0)$ to $(a, 0)$. (a) Consider the case of no circulation. (b) Consider the case of circulation chosen so that the flow attaches to the plate at $(a, 0)$. (c) Show that the flow past a flat plate airfoil with Kutta condition at angle of attack can be generated by the weighted superposition of the flow in part (b) and a uniform free stream in the positive x direction. Use plots of the streamlines and equipotentials to illustrate your solutions in each part. Worked example solution

Solution Problem 2.3

Potential and streamfunction are given by the real and imaginary parts of equation 2.7.30

$$w(z) = U_\infty z\cos\alpha - iU_\infty\sqrt{z^2-a^2}\sin\alpha - \frac{i\Gamma}{2\pi}\ln(\frac{z+\sqrt{z^2-a^2}}{2})$$

So, for a vertical free stream $\alpha = \pi/2$ and we have

$$\psi(z) = imag\left(-iU_\infty\sqrt{z^2-a^2} - \frac{i\Gamma}{2\pi}\ln\left(\frac{z+\sqrt{z^2-a^2}}{2}\right)\right)$$

And

$$\phi(z) = real\left(-iU_\infty\sqrt{z^2-a^2} - \frac{i\Gamma}{2\pi}\ln\left(\frac{z+\sqrt{z^2-a^2}}{2}\right)\right)$$

If necessary, these relations can be expanded by using the substitutions $r_1e^{i\theta_1} = z^2-a^2$ where

$$r_1 = \sqrt{(x^2-y^2-a^2)^2+(2xy)^2}$$

$$\theta_1 = \arctan\left(\frac{2xy}{x^2-y^2-a^2}\right)$$

and $r_2e^{i\theta_2} = z+\sqrt{z^2-a^2}$

$$r_2 = \sqrt{\left(x+\sqrt{r_1}\cos\theta_1/2\right)^2+\left(y+\sqrt{r_1}\sin\theta_1/2\right)^2}$$

$$\theta_2 = \arctan\left(\frac{y+\sqrt{r_1}\sin\theta_1/2}{x+\sqrt{r_1}\cos\theta_1/2}\right)$$

Giving

$$\psi(z) = -U_\infty\sqrt{r_1}\cos\frac{\theta_1}{2} - \frac{\Gamma}{2\pi}\ln\frac{r_2}{2}$$

$$\phi(z) = U_\infty\sqrt{r_1}\sin\frac{\theta_1}{2} + \frac{\Gamma}{2\pi}\theta_2$$

(a) For no circulation the result is

$$\psi(z) = imag\left(-iU_\infty\sqrt{z^2-a^2}\right)$$

$$\phi(z) = real\left(-iU_\infty\sqrt{z^2-a^2}\right)$$

Or

$$\psi(z) = -U_\infty\sqrt{r_1}\cos\frac{\theta_1}{2}$$

$$\phi(z) = +U_\infty\sqrt{r_1}\sin\frac{\theta_1}{2}$$

(b) To satisfy the Kutta condition we apply equation 2.7.31 for $\alpha = \pi/2$ so that $\Gamma = -2\pi a U_\infty$, and thus

$$\psi(z) = imag\left(-iU_\infty\sqrt{z^2 - a^2} + iU_\infty a \ln\left(\frac{z + \sqrt{z^2 - a^2}}{2}\right)\right)$$

$$\phi(z) = real\left(-iU_\infty\sqrt{z^2 - a^2} + iU_\infty a \ln\left(\frac{z + \sqrt{z^2 - a^2}}{2}\right)\right)$$

or

$$\psi(z) = -U_\infty\sqrt{r_1}\cos\frac{\theta_1}{2} + aU_\infty \ln\frac{r_2}{2}$$

$$\phi(z) = U_\infty\sqrt{r_1}\sin\frac{\theta_1}{2} - aU_\infty\frac{\theta_2}{2}$$

(c) For a horizontal free stream of velocity U we have $w_h(z) = Uz$

So, adding this to the vertical flow past the flat plate with free stream velocity V that satisfies the Kutta condition $w_v(z) = -iV\sqrt{z^2 - a^2} + iVa\ln\left(\frac{z+\sqrt{z^2-a^2}}{2}\right)$ we have

$$w(z) = Uz - iV\sqrt{z^2 - a^2} + iVa\ln\left(\frac{z + \sqrt{z^2 - a^2}}{2}\right)$$

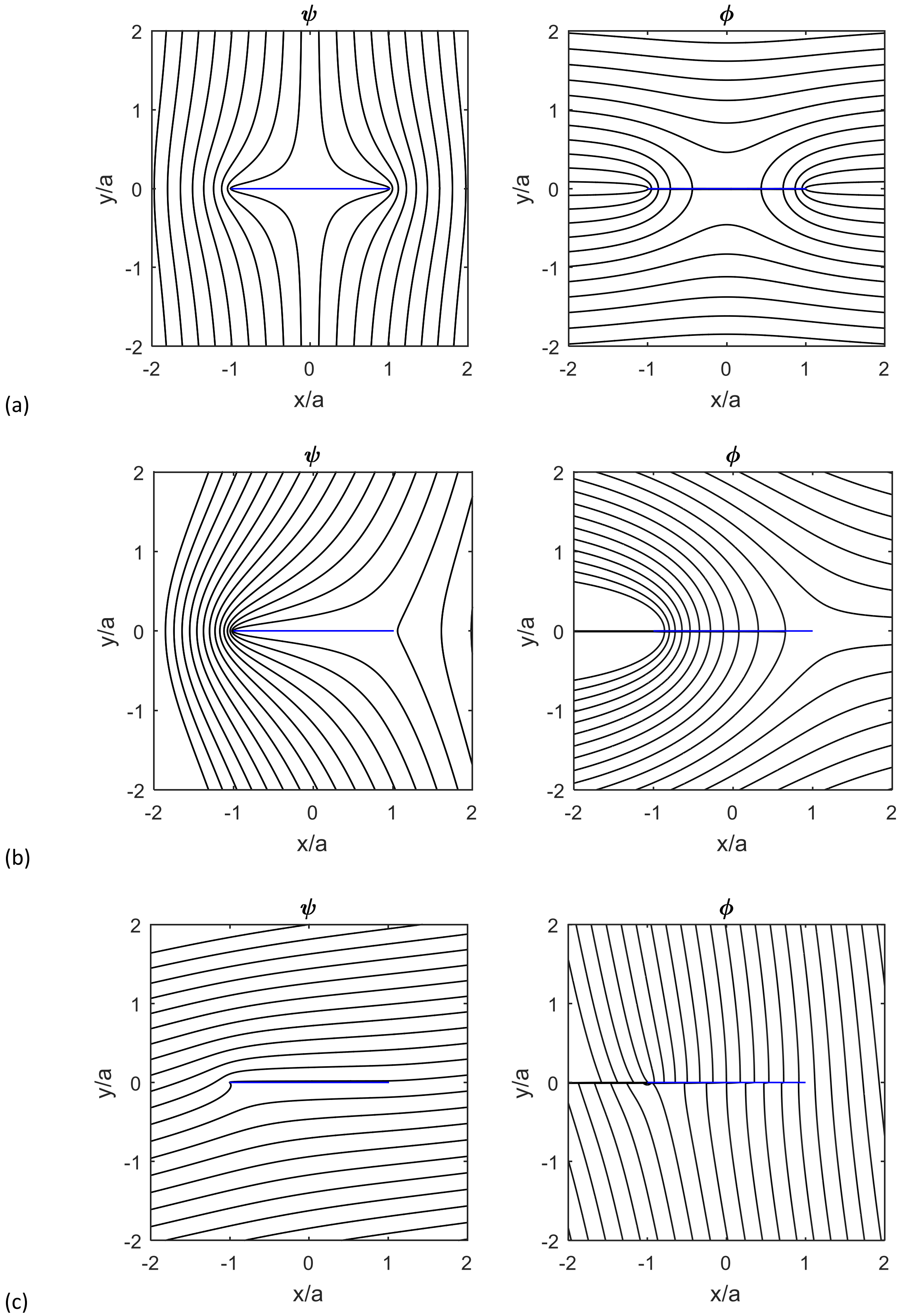
Letting $U = U_\infty \cos\alpha$ and $V = U_\infty \sin\alpha$ we get

$$w(z) = U_\infty \cos\alpha\, z - iU_\infty \sin\alpha\sqrt{z^2 - a^2} + iaU_\infty \sin\alpha \ln\left(\frac{z + \sqrt{z^2 - a^2}}{2}\right)$$

This matches equation 2.7.30 for angle of attack α with

$$\Gamma = -2\pi a U_\infty \sin\alpha$$

Which is the circulation needed to maintain the Kutta condition at angle of attack α. Plots are best done in Matlab using by computing the $w(z)$ functions in each case over a fine grid and contouring the real and imaginary parts. Note that (as discussed in appendix B) it is necessary to calculate $\sqrt{z^2 - a^2}$ as $\sqrt{z - a}\sqrt{z + a}$ to get the correct branch cut.



ψ
ϕ
y/a
x/a
2
1
0
-1
-2
(a)
(b)
(c)

Matlab code:

```
clear all; close all;
Uinf=1;a=1;
[x,y]=meshgrid(-2:.01:2,-2:.01:2);z=x+i*y;

%Part(a)
wa=-i*Uinf*sqrt(z-a).*sqrt(z+a);
psi=imag(wa);phi=real(wa);
figure(1);set(gcf,'position',[38    38   610   301]);
subplot(1,2,1);
contour(-2:.01:2,-2:.01:2,psi,-2.1:.2:2.1,'k');
hold on;plot([-a a],[0 0],'b');hold off %add plate
axis equal;xlabel('x/a');ylabel('y/a');title('\psi');
subplot(1,2,2);
contour(-2:.01:2,-2:.01:2,phi,-2.1:.2:2.1,'k');
hold on;plot([-a a],[0 0],'b');hold off %add plate
axis equal;xlabel('x/a');ylabel('y/a');title('\phi');

%Part(b)
wb=-i*Uinf*sqrt(z-a).*sqrt(z+a)+i*Uinf*a*log((z+sqrt(z-a).*sqrt(z+a))/2);
psi=imag(wb);phi=real(wb);
figure(2);set(gcf,'position',[38    38   610   301]);
subplot(1,2,1);
contour(-2:.01:2,-2:.01:2,psi,-2.1:.2:2.1,'k');
hold on;plot([-a a],[0 0],'b');hold off %add plate
axis equal;xlabel('x/a');ylabel('y/a');title('\psi');
subplot(1,2,2);
contour(-2:.01:2,-2:.01:2,phi,-2.1:.2:2.1,'k');
hold on;plot([-a a],[0 0],'b');hold off %add plate
axis equal;xlabel('x/a');ylabel('y/a');title('\phi');

%Part(c)
alpha=10*pi/180; %angle of attack
wc=Uinf*z;
w=wc*cos(alpha)+wb*sin(alpha)
psi=imag(w);phi=real(w);
figure(3);set(gcf,'position',[38    38   610   301]);
subplot(1,2,1);
contour(-2:.01:2,-2:.01:2,psi,-2.1:.2:2.1,'k');
hold on;plot([-a a],[0 0],'b');hold off %add plate
axis equal;xlabel('x/a');ylabel('y/a');title('\psi');
subplot(1,2,2);
contour(-2:.01:2,-2:.01:2,phi,-2.1:.2:2.1,'k');
hold on;plot([-a a],[0 0],'b');hold off %add plate
axis equal;xlabel('x/a');ylabel('y/a');title('\phi');
```