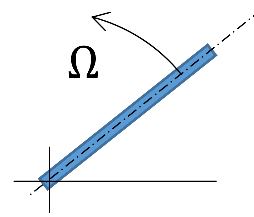
Chapter 5

5.1 As part of an air show, a jet airplane in straight and steady flight at Mach 0.4, passes by an observer at a distance of 300m. Assuming the jet has the directivity of a compact longitudinal quadrupole, aligned with the flight direction determine the variation in frequency and amplitude of the sound heard by the observer as a function of time. Normalize your answers based on the values heard when the airplane is at its closest approach. Plot your results in terms of the angle of the airplane to the observer from 20 to 160 degrees, with 90 degrees being defined at closest approach. Normalize the curves on their maximum values in this range. You may ignore the effect of the ground or any surrounding structures and consider the observer to be in the far field.
Worked example solution



5.2 Sound is made by a 5-m radius one-bladed rotor spinning in air with angular velocity Ω as shown. The rotor chordlength is small and so the rotor may be approximated as a straight line. The sound is heard by an observer located in the plane of the rotor at a distance of 20 m from the axis of rotation. (a) Find an expression for the retarded time τ in terms of observer time t, Ω and radial distance on the rotor from the hub R. Note that this expression is simple to write as explicit for t (which is fine for your answer) but difficult or impossible to write as explicit for τ. (b) Plot three curves on the same axes showing the retarded time as a function of observer time for the rotor blade tip, each normalized by multiplying by Ω, . The plots should cover one rotation and the three curves should correspond to tip Mach numbers of 0.3, 0.7 and 1.3.

5.3 The tip of a helicopter rotor blade of radius $R_o = 5m$ spinning at angular velocity Ω becomes damaged, and the resulting tip geometry produces a loud whistle. The sound source is determined to be sinusoidal force in the radial direction, of amplitude 7 Newtons, produced on the blade tip. With the helicopter in hover the tip Mach number is 0.5 the frequency of the force fluctuation is 5kHz. We are concerned with determining the sound heard by an observer 200m away from the rotor hub in the plane of the rotor. An engineer proposes using equation 5.3.1 to solve this problem:

$$p'(\mathbf{x},t) = -Re\left\{\frac{\partial}{\partial x_i}\left[\frac{\hat{F}_i e^{-i\omega_o \tau}}{4\pi r|1-M_r|}\right]_{\tau=\tau^*}\right\}$$

(a) Does this equation assume that the noise generating portion of the blade is acoustically compact? If so, explain exactly how this assumption is used in formulating this equation.

(b) One route to solving the above equation is to use the relationship between the observer divergence and the observer time derivative.

$$\frac{\partial}{\partial x_i} = -\frac{x_i}{|\mathbf{x}|}\frac{1}{c_\infty}\frac{\partial}{\partial t}$$

Explain the physical basis of this relation, and justify its quantitative form.

(c) Find an expression for $\hat{F}_i$ in terms of Ω and source time. Determine M_r as a function of source time, making any appropriate simplifications. Evaluate Ω for the conditions given above.

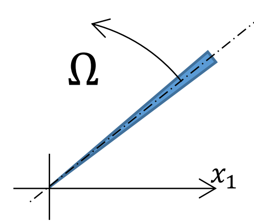
(d) Give the relationship that is best to use to give source time as a function of observer time. Starting with this relation explain the mathematical origin of the Doppler shift in frequency. What will be the maximum and minimum frequencies heard by the observer in this case?

(e) Another route to solving the above equation will involve determining at some stage the derivative $\partial\big[\hat{F}_i\big]_{\tau=\tau^*}/\partial x_i$ (amongst others). What is $\partial\big[\hat{F}_i\big]_{\tau=\tau^*}/\partial x_i$ in terms of $\mathbf{x}$, M_r, Ω and c_∞?

5.4. Consider the sound is made by a one-bladed rotor of radius $R_o = 5\,m$ spinning in air with angular velocity Ω as shown, as heard by an observer at $x_i = (20\,m, 0, 0)$. Verify that the relationship between observer and source time for the blade planform centerline is,

$$\Omega\tau = \Omega t - M_{tip}\sqrt{\left(\frac{x_1}{R_o} - \frac{R}{R_o}\cos(\Omega\tau)\right)^2 + \left(\frac{R}{R_o}\right)^2 \sin^2(\Omega\tau)}$$

where $M_{tip} = \Omega R_o / c_\infty$. Reformulate this expression to give, as a function of source time, the radial position on the blade centerline that emits the sound heard by the observer at t. Write a short Matlab code that implements this relationship as a function giving this radius R/R_o vs the blade orientation $\Omega\tau$ (i.e the effective blade centerline shape as heard by the observer). The function inputs should be x_1, R_o, Ωt and M_{tip}. Be careful to ensure that the blade positions are consistent with reality and that your $\Omega\tau$ values refer to the correct portion of the blade rotation. Use your function to plot the apparent blade centerline shape at $M_{tip} = 0.3$, and 0.7 for $\Omega t = \pi/2$



Solution Problem 5.1

To solve this problem, we begin with the 5.2.15 dropping all but the quadrupole term and writing in terms of the acoustic pressure

$$p'(\mathbf{x},t) = \frac{x_i x_j}{|\mathbf{x}|^2}\frac{1}{c_\infty^2}\frac{\partial^2}{\partial t^2}\int_{V_o}\left[\frac{T_{ij}}{4\pi|\mathbf{x}||1-M_r|}\right]_{\tau=\tau^*} dV(\mathbf{z})$$

For a truly compact source

$$p'(\mathbf{x},t) = \frac{x_i x_j}{|\mathbf{x}|^2}\frac{1}{c_\infty^2}\frac{1}{4\pi|\mathbf{x}||1-M_r|}\ \frac{\partial^2}{\partial t^2}\int_{V_o}\left[T_{ij}\right]_{\tau=\tau^*} dV(\mathbf{z})$$

Or, using equation 5.2.16

$$p'(\mathbf{x},t) = \frac{x_i x_j}{|\mathbf{x}|^2}\frac{1}{c_\infty^2}\frac{1}{4\pi|\mathbf{x}||1-M_r|(1-M_r)^2}\ \left[\int_{V_o}\frac{\partial^2 T_{ij}}{\partial \tau^2} dV(\mathbf{z})\right]_{\tau=\tau^*}$$

The sound field for by a compact longitudinal quadrupole in a quiescent environment is given by equation 3.7.5, which we re-write as

$$p'_{QUAD}(\mathbf{x},t) = Re\left\{\left(\frac{x_1}{|\mathbf{x}|}\right)^2\frac{1}{c_\infty^2}\ \frac{1}{4\pi|\mathbf{x}|}\hat{A}e^{-i\omega\tau}e^{ik|\mathbf{x}|}\right\}$$

Comparing these two equations, we see that

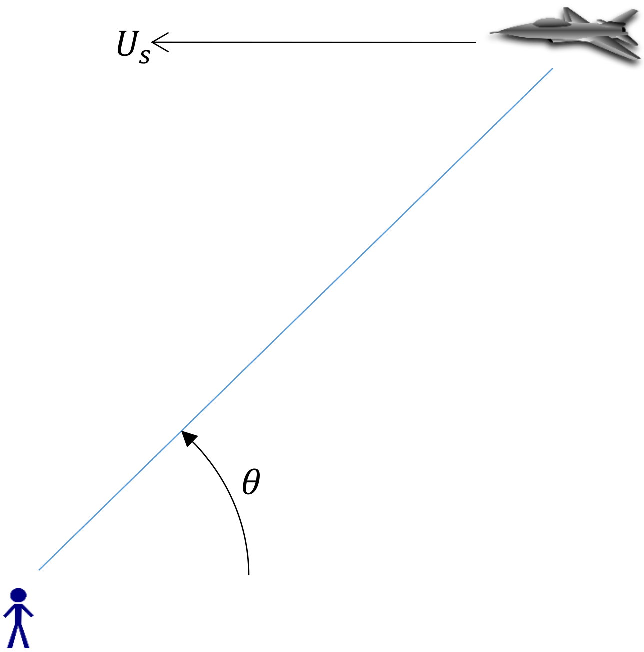
$$\int_{V_o}\frac{\partial^2 T_{ij}}{\partial \tau^2} dV(\mathbf{z})\ \equiv \delta_{i1}\delta_{j1}Re\left\{\hat{A}e^{-i\omega\tau}e^{ik|\mathbf{x}|}\right\}$$

Thus the noise of our aircraft is given by

$$\begin{aligned} p'(\mathbf{x},t) &= \frac{x_i x_j}{|\mathbf{x}|^2}\frac{1}{c_\infty^2}\frac{1}{4\pi|\mathbf{x}||1-M_r|(1-M_r)^2}\ \ Re\left[\delta_{i1}\delta_{j1}\hat{A}e^{-i\omega\tau}e^{ik|\mathbf{x}|}\right]_{\tau=\tau^*} \\ &= \frac{x_1^2}{|\mathbf{x}|^2}\frac{1}{c_\infty^2}\frac{1}{4\pi|\mathbf{x}||1-M_r|(1-M_r)^2}\ Re\left[\hat{A}e^{-i\omega\tau}e^{ik|\mathbf{x}|}\right]_{\tau=\tau^*} \end{aligned}$$

To determine the retarded time we have to consider the kinematics of the problem which are illustrated below

Considering a coordinate system origin fixed at the point in space where the aircraft is located at the instant shown, then the relationship for τ^* is given by equation 5.3.3, and thus the far-field approximation is given by equation 5.3.4



$$\tau = \tau^* = \frac{t - |\mathbf{x}|/c_\infty}{1 - M_r}$$

Thus the far-field sound becomes

$$p'(\mathbf{x}, t) = \frac{x_1^2}{|\mathbf{x}|^2} \frac{1}{c_\infty^2} \frac{1}{4\pi|\mathbf{x}||1 - M_r|(1 - M_r)^2} Re\left\{\hat{A}e^{-i\omega\left(\frac{t-|\mathbf{x}|/c_\infty}{1-M_r}\right)} e^{ik|\mathbf{x}|}\right\}$$

Giving the source zero absolute phase shift then this becomes

$$p'(\mathbf{x}, t) = \frac{x_1^2}{|\mathbf{x}|^2} \frac{1}{c_\infty^2} \frac{1}{4\pi|\mathbf{x}||1 - M_r|(1 - M_r)^2} A\cos\left(k|\mathbf{x}| - \omega\left(\frac{t - |\mathbf{x}|/c_\infty}{1 - M_r}\right)\right)$$

The sound frequency therefore varies due to Doppler shift as

$$\frac{1}{1 - M_r} = \frac{1}{1 - M_s\cos\theta}$$

where M_s is the aircraft Mach number. Likewise, the amplitude varies as

$$\frac{x_1^2}{|\mathbf{x}|^2} \frac{1}{|\mathbf{x}||1 - M_r|(1 - M_r)^2} = \frac{\cos^2\theta}{(1 - M_s\cos\theta)^3|\mathbf{x}|}$$

Where the magnitude has been dropped since $M_r < 1$. These functions are plotted using the Matlab script below. Note that by using the result in this way (with variable aircraft position) we are essentially re-defining the origin of **x** at each point along the flight path.

```
clear all;close all;

th=[20:160]*pi/180;
m=0.4;
fm=1./(1-m*cos(th));
x2=300;x=x2./sin(th);
am=cos(th).^2./(1-m*cos(th)).^3./x;

figure
plot(th*180/pi,fm/max(fm),'k-',th*180/pi,am/max(am),'b-');
legend('Frequency modulation','Amplitude modulation');
xlabel('\theta (deg.)');
```