Chapter 6

6.1 Consider two-dimensional ideal flow over a circular cylinder without circulation of unit radius, with free stream velocity U_∞ in the x_1 direction. (a) Write down analytic expressions for the streamfunction and velocity magnitude as functions of position, and (b) analytical expressions for the drift functions X_2 and X_3. (c) Write a short Matlab program to determine and plot contours of the drift functions X_1 and X_2 for X_1 and $X_2 = -2$ to 2 as functions of position (x_1, x_2) for x_1 and x_2=-2 to 2.
Worked Example Solution

6.2 An initially uniform mean flow of velocity U_∞ containing turbulence flows over a two dimensional circular cylinder of unit radius. Consider the flow on the stagnation streamline upstream of the cylinder, where the mean flow is accurately described by the ideal flow solution for the cylinder without circulation. (a) Determine expressions for the drift function gradients as function of distance along the streamline. (b) Determine the displacement vectors as a function of this distance. (c) Determine the complex amplitude of the vorticity vector for the portion of the turbulence initially at wavevector $\mathbf{k}$, in terms of its undistorted amplitude and the drift function, as a function of this distance. (d) Determine the fluctuating velocity vector of this same portion**,** in terms of the initial vorticity amplitude as a function of this distance. (e) Continuing from part (c), what is the actual wave vector of the turbulence as a function of this distance? (f) What assumptions must hold for your answers in parts (c) through (e) to be accurate?

6.3. The flow of a free stream over the nose of a two dimensional body is accurately described by the ideal flow complex velocity

$$w'(z) = U_\infty + \frac{U_\infty d}{2\pi z}$$

where $d = 0.5\ m$.
(a) Write down analytic expressions for the streamfunction and velocity magnitude as functions of position.
(b) Write down analytic expressions for the drift functions X_2 and X_3.
(c) Write a short Matlab program to determine and plot contours of the drift functions X_1 and X_2 for X_1 and $X_2 = -0.6$ to 0.6 as functions of position (x_1, x_2) for x_1 and x_2=-0.6 to 0.6.
(d) Extend your code to determine and plot the variation in relative displacement vector lengths $|\delta\mathbf{l}^{(1)}/h_1|$ and $|\delta\mathbf{l}^{(2)}/h_2|$ along a streamline initiated in the freestream at $x_2 = 0.2$ as a function of x_1 for x_1 =-0.6 to 0.6

6.4 Use the definition of the wavenumber transform to write Equation 6.5.5 (an expression for the sound radiated from a flat plate airfoil in a moving medium) in terms of $\Delta\tilde{\tilde{p}}\left(k_1^{(o)}, k_3^{(o)}, \omega\right)$. Define $k_1^{(o)}$ and $k_3^{(o)}$ in your solution.

Worked Example Solution

Solution Problem 6.1

Solution

(a) Following equation 2.7.16 the complex velocity and complex potential for ideal flow past a unit radius circular cylinder of velocity U_∞ at an angle $\alpha = 0$, are

$$w'(z) = U_\infty - U_\infty/z^2$$

$$w(z) = U_\infty z + U_\infty/z$$

where we are placing the origin at the center of the cylinder ($z_1 = 0$). The streamfunction is the imaginary part of the complex potential, and so since $z = x_1 + ix_2$ and $1/z = z^*/|z|^2 = (x_1 - ix_2)/r^2$ where $r^2 = x_1^2 + x_2^2$, then

$$\psi = U_\infty x_2 - U_\infty x_2/r^2$$

Noting that $1/z^2 = {z^*}^2/|z|^4 = (x_1^2 + x_2^2 - 2ix_1x_2)/|z|^4 = (r^2 - 2ix_1x_2)/r^4$

We see that

$$|w'(z)|^2 = |\mathbf{U}|^2 = U^2 = U_\infty^2\left(1 - \frac{1}{r^2}\right)^2 + U_\infty^2\left(\frac{2x_1x_2}{r^4}\right)^2$$

(b) Surfaces of constant X_2 and X_3 are streamsurfaces of the flow, but X_2 and X_3 must become x_2 and x_3 at upstream infinity. This works in the spanwise direction if, simply,

$$X_3 = x_3$$

and in the cross stream direction if $X_2 = \psi/U_\infty$ so that,

$$X_2 = x_2 - x_2/r^2$$

(c) To get X_1 we need to integrate

$$X_1 = U_\infty \int_{streamline} \frac{d\sigma}{U}$$

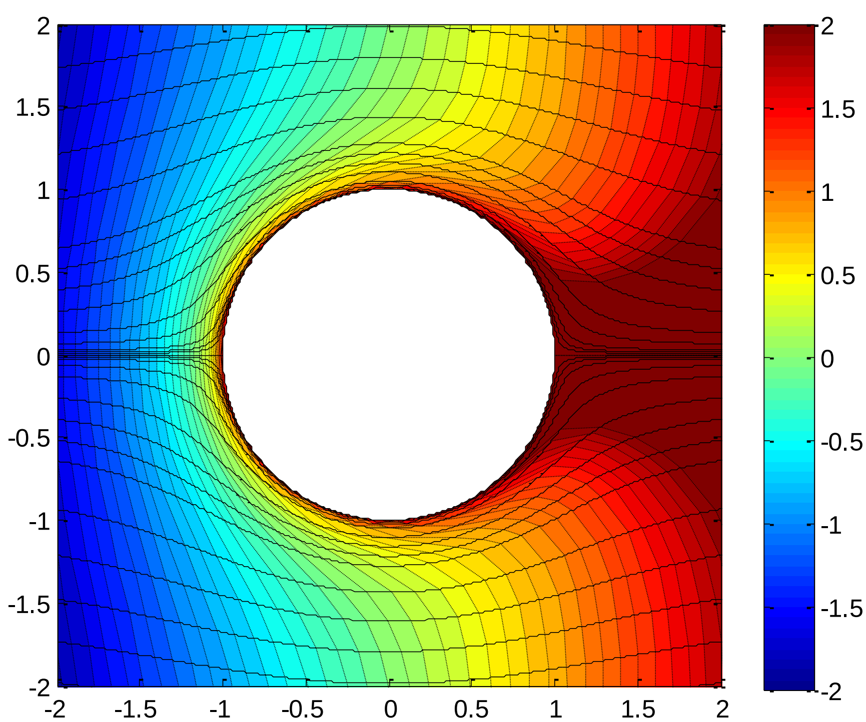
This can be done numerically as part of a Matlab code for calculating the streamlines (contours of X_2) by directly integrating $w'(z)$. Note the uneven spacing of the streamlines in the following code in order to sufficiently define the functions in stagnation region of the cylinder. The color scale in the plot corresponds to X_1.

Matlab code:

```
clear all;close all;
X2=[-2:.25:-.5 -.4:.1:-.1 -.05 -.02 -.01 -.005 0 0.005 0.01 0.02 0.05
.1:.1:.4 .5:.25:2];
step=.01;maxstep=3000;
for n=1:length(X2)
    nstep=0;z=-10+i*X2(n);X1(1,n)=-10;
    while nstep<maxstep
        w=1-1/z^2;
        z1=z+conj(w)*step;
        w1=1-1/z1^2;
        z=z+conj(w+w1)*step/2;
        nstep=nstep+1;
        x1(nstep,n)=real(z);x2(nstep,n)=imag(z);
    end
end
X1=repmat([1:3000]'*step-10,[1 length(X2)]);

figure
contourf(x1,x2,X1,[-2:.1:2],'k:'); hold on;
plot(x1,x2,'k-');
fill(cos([0:360]*pi/180),sin([0:360]*pi/180),'w');
axis image;
xlim([-2 2]);ylim([-2 2]);
colorbar
```

Plot:



Solution Problem 6.4

Equation 6.5.5 is

$$\tilde{p}(\boldsymbol{x},\omega)c_\infty^2 \approx \frac{-i\omega x_2 e^{ik_o r_e}}{4\pi c_\infty r_e^2}\int_{-c/2}^{c/2}\int_{-b/2}^{b/2}\Delta\tilde{p}(\boldsymbol{y},\omega)e^{-\frac{ik_o x_1 y_1}{r_e}-\frac{ik_o x_3 y_3 \beta^2}{r_e}-ik_o M(x_1-y_1)}dy_1 dy_3$$

The limits of the integral can be changed since the pressure jump is zero for all points not on the airfoil so the right hand side becomes, with rearrangement,

$$\approx \frac{-i\omega x_2 e^{ik_o r_e}}{4\pi c_\infty r_e^2}(2\pi)^2\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Delta\tilde{p}(\boldsymbol{y},\omega)e^{-i\left(\frac{k_o x_1}{r_e}-k_o M\right)y_1-i\left(\frac{k_o x_3 \beta^2}{r_e}\right)y_3}e^{-ik_o M x_1}dy_1 dy_3$$

Rearranging

$$\approx \frac{-i\pi\omega x_2 e^{ik_o r_e-ik_o M x_1}}{c_\infty r_e^2}\left[\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Delta\tilde{p}(\boldsymbol{y},\omega)e^{-ik_1^{(o)}y_1-ik_3^{(o)}y_3}dy_1 dy_3\right]$$

Where

$$k_1^{(o)} = k_o\left(\frac{x_1}{r_e}-M\right) \qquad k_3^{(o)} = k_o\beta^2\left(\frac{x_3}{r_e}\right)$$

Definition of the wavenumber transform

$$\Delta\tilde{\tilde{p}}\left(k_1^{(o)},k_3^{(o)},\omega\right) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Delta\tilde{p}(\boldsymbol{y},\omega)e^{-ik_1^{(o)}y_1-ik_3^{(o)}y_3}dy_1 dy_3$$

Substituting this into the above equation

$$\approx \frac{-i\pi\omega x_2 e^{ik_o r_e-ik_o M x_1}}{c_\infty r_e^2}\Delta\tilde{\tilde{p}}\left(k_1^{(o)},k_3^{(o)},\omega\right)$$

Where

$$k_1^{(o)} = k_o\left(\frac{x_1}{r_e}-M\right) \qquad k_3^{(o)} = k_o\beta^2\left(\frac{x_3}{r_e}\right)$$