Chapter 7

7.1 (a) Prove that Howe's result, equation 7.3.4, is consistent with the Kutta-Joukowski theorem, equation 2.7.23, by considering first a line vortex of strength Γ, parallel to x_3, located at $x_2 = h$, and traveling in the x_1 direction at speed U

(b) Consider a two-dimensional circular cylinder centered on the x_3 axis of radius R. Determine the unsteady force components per unit span exerted by the cylinder on the fluid, as functions of time, due to the passage of the above line vortex in the x_1 direction at speed U. Assume there is no circulation generated around the cylinder.

(c) Plot the time trace of the force components, normalized on $\rho_o U\Gamma$ as a function of normalized time tU/R from -2 to 2, for $h/R = 1.5$.

(d) What equation would you use to determine the far field sound radiated by a small acoustically compact spanwise portion of the cylinder?

Worked example solution

Solution Problem 7.1

(a) Equation 7.3.4 gives the force applied to the fluid by a body as

$$F_i = -\rho_o \int\limits_V \nabla Y_i . (\boldsymbol{\omega} \times \mathbf{v})\, dV$$

Consider a line vortex of strength Γ parallel to the x_3 axis located at $x_2 = h$ and traveling along the x_1 axis at speed U. For this we have

$$\boldsymbol{\omega} = \Gamma \delta(x_1 - Ut)\delta(x_2 - h)\mathbf{e_3}$$

where $\mathbf{e_i}$ denotes the unit vector in the direction of x_i. The velocity of the vortex $\mathbf{v} = U\mathbf{e_1}$, and so

$$(\boldsymbol{\omega} \times \mathbf{v}) = U\Gamma \delta(x_1 - Ut)\delta(x_2 - h)\mathbf{e_2}$$

Clearly, therefore, the vortex can only generate a force in the x_2 direction. Consider thus the Kirchhoff coordinate for a unit free stream in the x_2 direction, which will give $\nabla Y_2 = \mathbf{e_2}$, and so

$$F_2 = -\rho_o \int\limits_V U\Gamma \delta(x_1 - Ut)\delta(x_2 - h)\, dV$$

and so the force per unit span in the x_3 direction is

$$f_2 = -\rho_o U\Gamma$$

Now, if we change our frame of reference to be traveling with the vortex then we will see a free stream in the x_1 direction of velocity $U_\infty = -U$. Therefore, in terms of this free stream velocity, we have

$$f_2 = \rho_o U_\infty \Gamma$$

This recall is the force on the fluid. Thus the force on the vortex must be,

$$f_2 = -\rho_o U_\infty \Gamma$$

(b) For this problem we need to determine the Kirchhoff coordinate gradient ∇Y_i for the flow in the x_i direction around the cylinder. To get both F_1 and F_2 components we will have to consider $i = 1,2$. The complex velocity for acyclic flow past a circular cylinder is given by equation 2.7.16,

$$w'(z) = e^{-i\alpha} - \frac{R^2 e^{i\alpha}}{z^2}$$

Where we have taken $U_\infty = 1$ and where $\alpha = 0$ for $i = 1$ and $\alpha = \pi/2$ for $i = 2$. Now,

$$\nabla Y_i = Re\{w'(z)\}\mathbf{e_1} - Im\{w'(z)\}\mathbf{e_2}$$

So,

$$\nabla Y_i . (\boldsymbol{\omega} \times \mathbf{v}) = -Im\{w'(z)\}U\Gamma \delta(x_1 - Ut)\delta(x_2 - h)$$

Now,

$$Im\{w'(z)\} = -\sin\alpha - Im\left\{\frac{R^2(\cos\alpha + i\sin\alpha)(x_1^2 - x_2^2 - 2ix_1x_2)}{r^4}\right\}$$

$$= -\sin\alpha - \frac{R^2}{r^4}[(x_1^2 - x_2^2)\sin\alpha - 2x_1x_2\cos\alpha]$$

where $r^2 = x_1^2 + x_2^2$. So,

$$f_i = -\rho_o U\Gamma\left\{\sin\alpha + \frac{R^2}{(U^2t^2 + h^2)^2}[(U^2t^2 - h^2)\sin\alpha - 2Uth\cos\alpha]\right\}$$

So,

$$f_1 = \rho_o U\Gamma\frac{2UthR^2}{(U^2t^2 + h^2)^2}$$

and

$$f_2 = -\rho_o U\Gamma\left\{1 + \frac{R^2(U^2t^2 - h^2)}{(U^2t^2 + h^2)^2}\right\}$$

(c) The above expressions are normalized as

$$\frac{f_1}{\rho_o U\Gamma} = \frac{2\frac{Ut}{R}\frac{h}{R}}{\left(\left[\frac{Ut}{R}\right]^2 + \left[\frac{h}{R}\right]^2\right)^2}$$

and

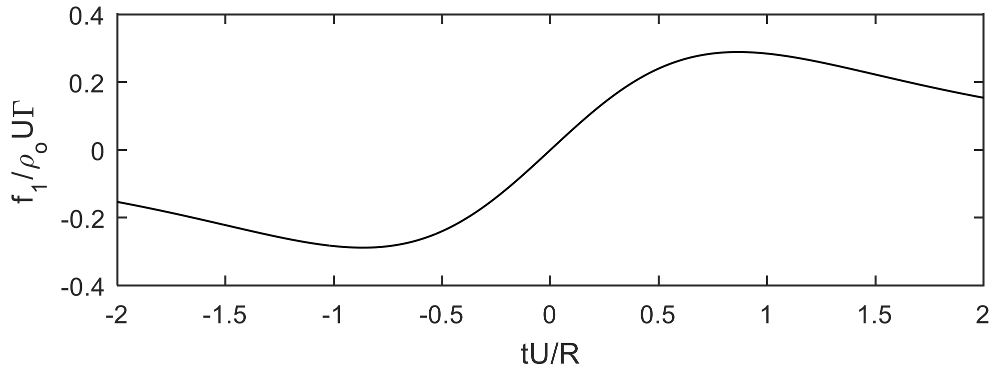
$$\frac{f_2}{\rho_o U\Gamma} = -1 - \frac{\left(\left[\frac{Ut}{R}\right]^2 - \left[\frac{h}{R}\right]^2\right)}{\left(\left[\frac{Ut}{R}\right]^2 + \left[\frac{h}{R}\right]^2\right)^2}$$

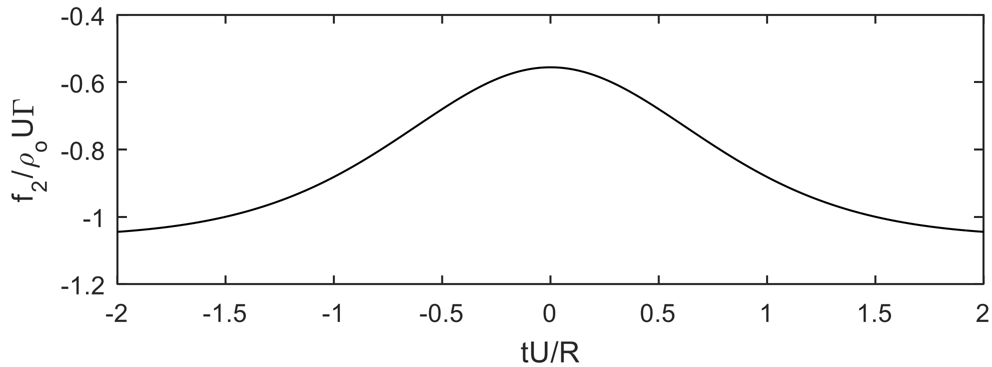
Plotting code and plot:

```
clear all; close all;
tn=-2:.01:2;
hR=1.5;

f1n=2*tn*hR./(tn.^2+hR.^2).^2;
f2n=-1-(tn.^2-hR.^2)./(tn.^2+hR.^2).^2;

figure
subplot(2,1,1)
plot(tn,f1n,'k-');
xlabel('tU/R');ylabel('f_1/\rho_oU\Gamma');
subplot(2,1,2)
plot(tn,f2n,'k-');
xlabel('tU/R');ylabel('f_2/\rho_oU\Gamma');
```





(d) Equation 4.4.7

$$p' = \frac{x_i}{4\pi|\mathbf{x}|^2 c_\infty}\frac{\partial F_i}{\partial t}$$