AEROACOSTICS OF LOW MACH NUMBER FLOWS

Short Course

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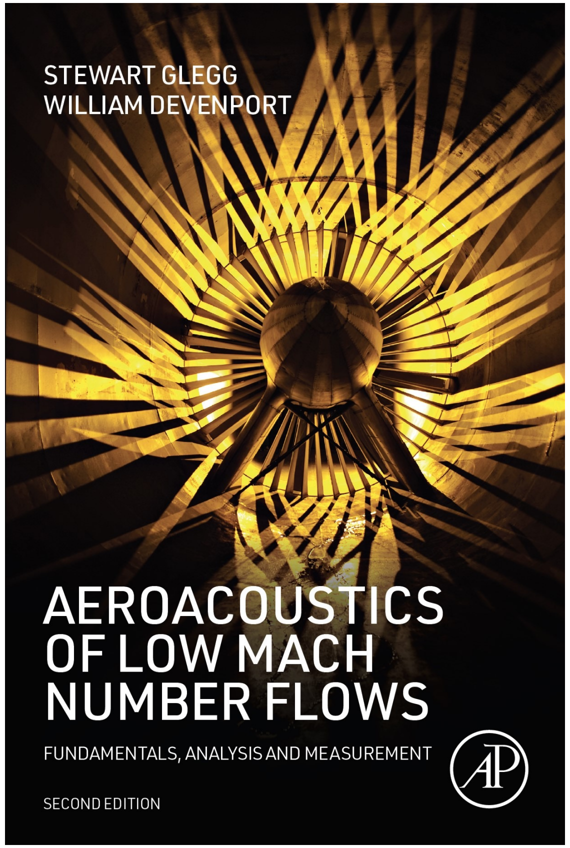
Rome, June 3rd 2024

Venue: Department of Civil, Computer Science and Aeronautical Technologies Engineering, in Via Vito Volterra 60 Classroom N14.



Course Objectives and Textbook

- *Addresses in detail sound from rotating blades, ducted fans, airframes, boundary layers, and more*
- *Presents theory in such a way that it can be used in computational methods and calculating sound levels*

Textbook

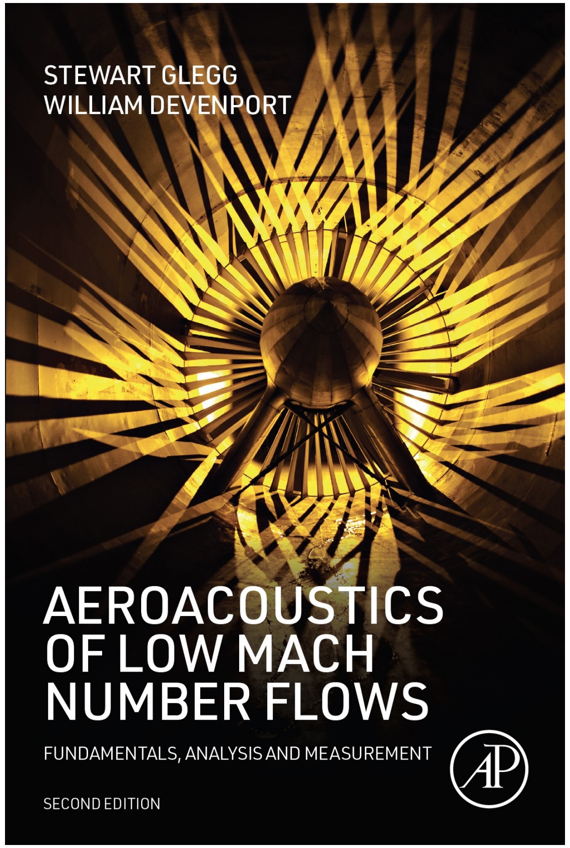
Aeroacoustics of Low Mach Number Flows:

Fundamentals, Analysis and Measurement

2nd Edition, Academic Press, 2023

Includes all the course material and problems sets.

There are also sample MATLAB codes, and experimental data that can be found at www.aeroacoustics.net.



Course Lectures and Arrangements

9 am	**Introduction to Course**
9.15 am	**Introduction to Aeroacoustics**
9.45 am	**Fundamentals and Linear Acoustics (Chapter 2 & 3)**
10.45 am	**Discussion**
11 am	**Break**
11.15 am	**Lighthill's Acoustic Analogy (Chapters 4 & 5)**
12.45 pm	**Discussion**
1 pm	**Lunch**
2.30 pm	**Turbulent Flows (Chapters 10,11, & 12)**
3.45 pm	**Discussion**
4.00 pm	**Break**
4.15 pm	**Propeller and Open Rotor Noise (Chapters 6 & 7)**
6.00 pm	**Discussion**
6.30 pm	**Adjourn**

Why are we here?

Why is Aeroacoustics Important?

- **16 million commercial airline flights tracked by the FAA each year**
- **This is only possible because of the development of the jet engine in the 1940’s**
- **It was realized at the outset that the limiting factor on commercial aircraft operations was the noise on take off and landing**
- **This is still one of the limiting factors for supersonic commercial flights, but there has been massive noise reductions in subsonic aircraft**

The Pioneers

- In the early 1950's a team at NACA in the US led by Harvey Hubbard, and a team in the UK led by E.J. Richards started to address the jet noise problem
- Measurements of jet noise were made at NACA facilities and by Lilley and Westley at Cranfield in the UK
- At the same time James Lighthill developed the Theory of Aerodynamic Noise which has laid the foundations for the topic
- This has guided jet and fan noise design ever since and we now have "quiet" aircraft that make over 25 dB less noise than their predecessors on take off.

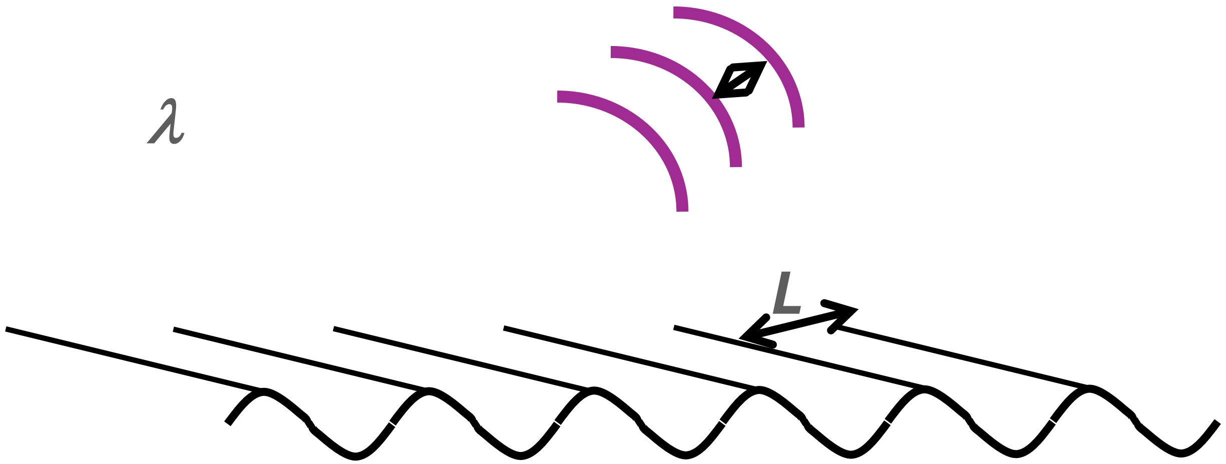
Sir James Lighthill

1924 to 1998

https://en.wikipedia.org/wiki/James_Lighthill

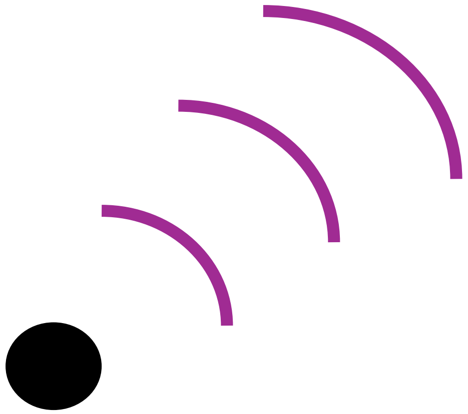
What causes sound?

Vibrating surfaces



Vibration of a large surface

$L>\lambda$

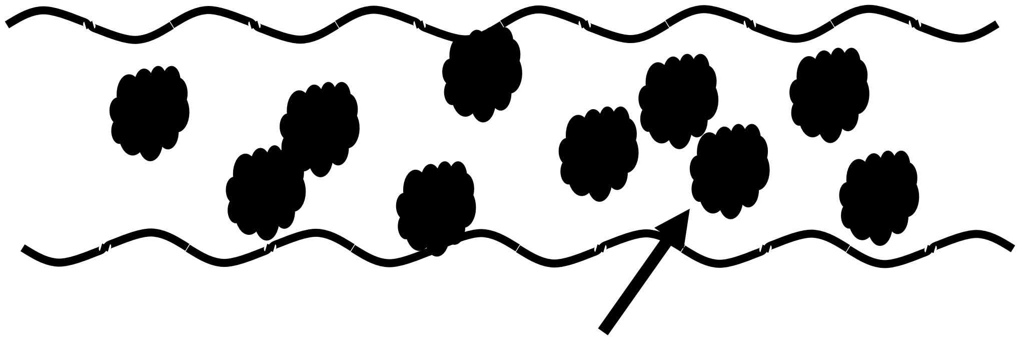


Bubble or a pulsating sphere

Requires mass injection

(1) Sound from Turbulence

Simple Jet Flow with No Solid Surfaces



Turbulent Eddies

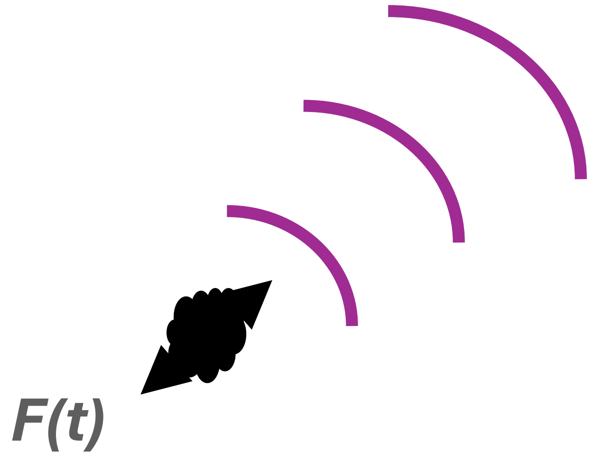
L<<λ

Eddies do not cause mass injection

So what is the mechanism???

(2) Sound from Turbulence

Force applied to Fluid

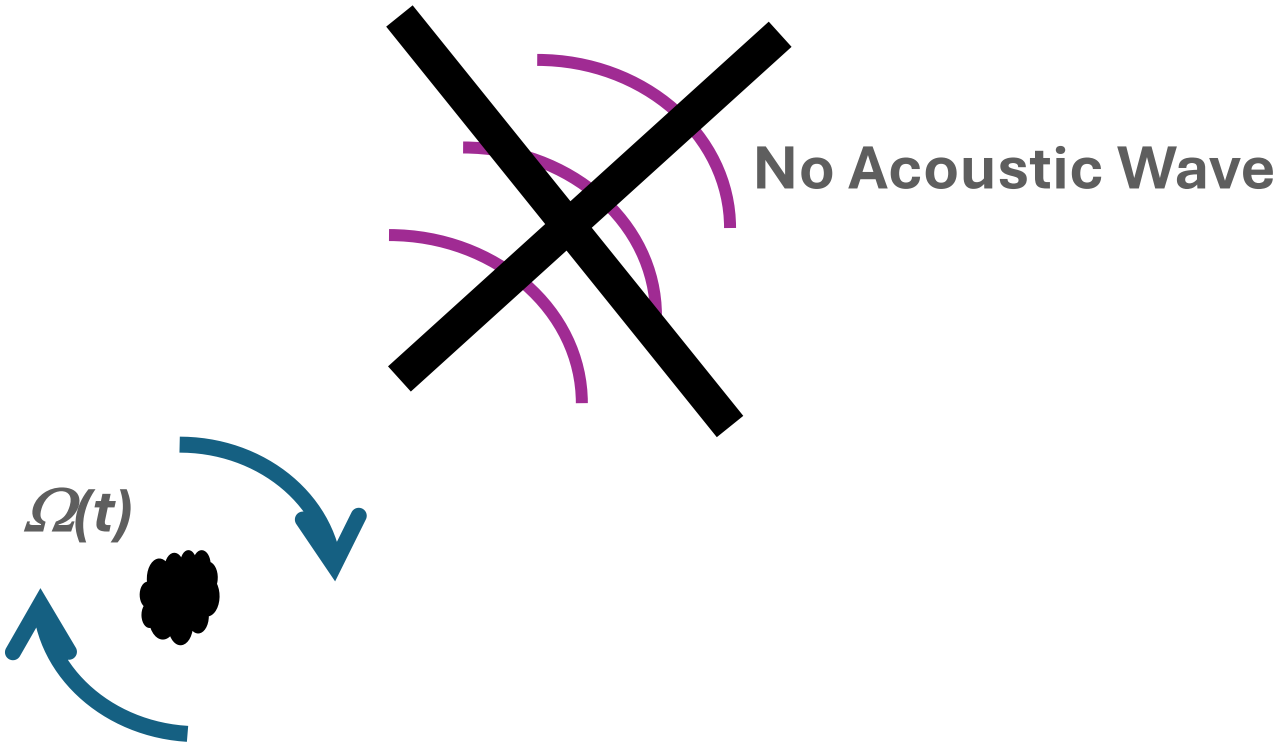


Net Force applied to a turbulent eddy *F(t)*

Requires body force like gravity or a magnetic force (or a surface)

(3) Sound from Turbulence

Vorticity



Rotating Eddies are orthogonal to outgoing Acoustic waves

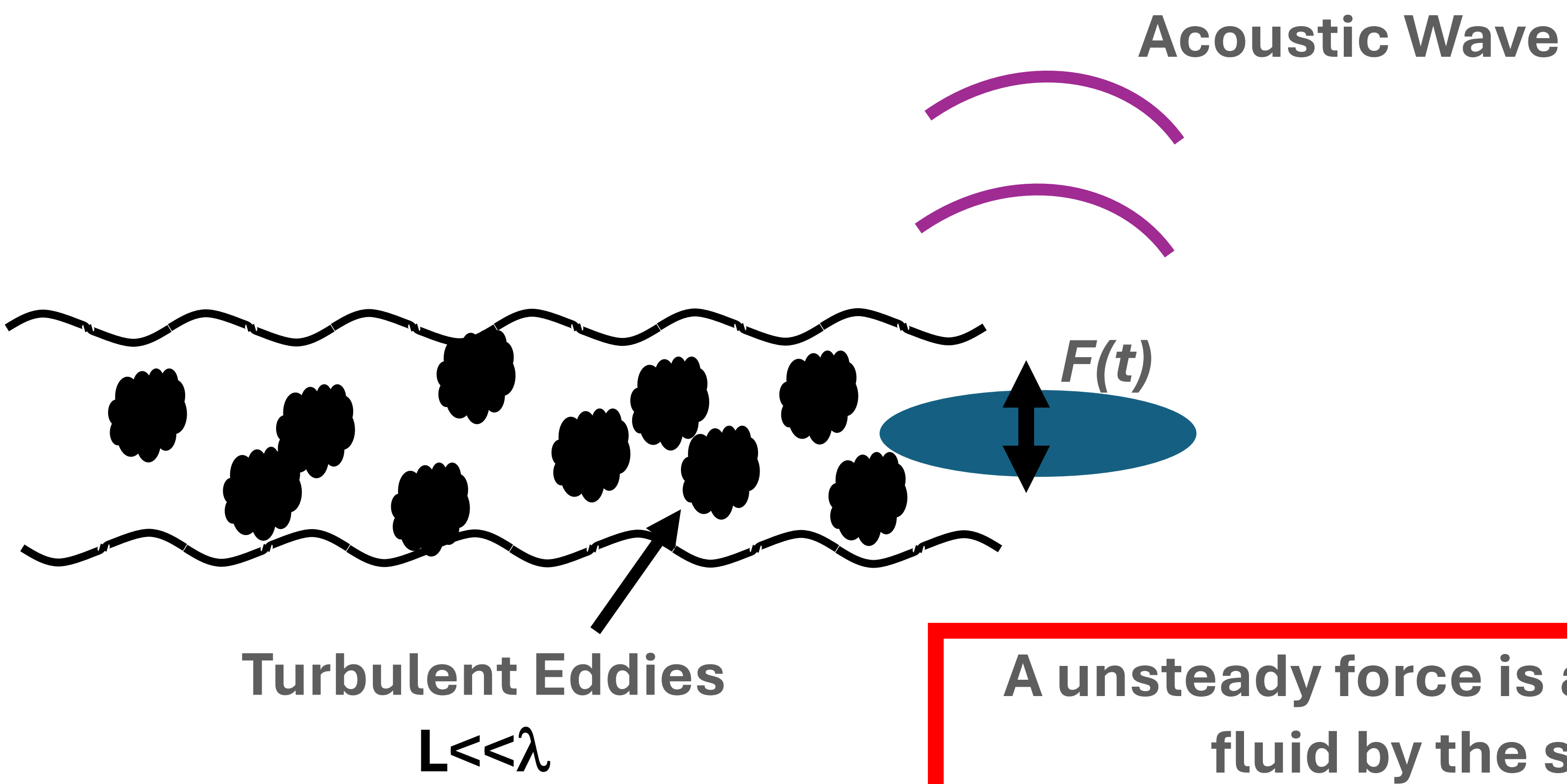
(4) Sound from Turbulence

Squishing Motion

This is Lighthill's Quadrupole Source

(5) Sound from Turbulence

Surfaces in the Flow



A unsteady force is applied to the fluid by the surface
Or
The fixed surface forces the fluid to move from it's undisturbed path

Linear Acoustics Module

Outline of Topics

- **Continuity and Momentum**
- **Compressibility**
- **Linearization Assumptions**
- **The Wave Equation**
- **Simple Boundary Conditions-the monopole source**
- **Superposition and the acoustic far field**
- **Dipole source motion**
- **Quadrupole source motion**

Notation

- Position $\mathbf{x} = (x_1, x_2, x_3) = x_i \quad i = 1, 2, 3$

  When we are talking about flows producing noise at a distant observer we will always use **x** to denote the position of the observer and **y** to denote the position within the flow (the 'source' of sound)

- Velocity $\mathbf{v} = (v_1, v_2, v_3) = v_i \quad i = 1, 2, 3$

  We will decompose the velocity into its mean (**U**) and fluctuating (**u**) parts when necessary, i.e.

  v=U+u

Summation subscript notation

Multiplicative terms with a repeated subscript, say i, is summed for $i = 1$ to 3

Dot product: $\mathbf{q}.\mathbf{v} = q_1 v_1 + q_2 v_2 + q_3 v_3 = \sum_{i=1}^{3} q_i v_i = q_i v_i$

Likewise $|\mathbf{x}|^2 = x_i^2$ (since squaring counts as repetition)

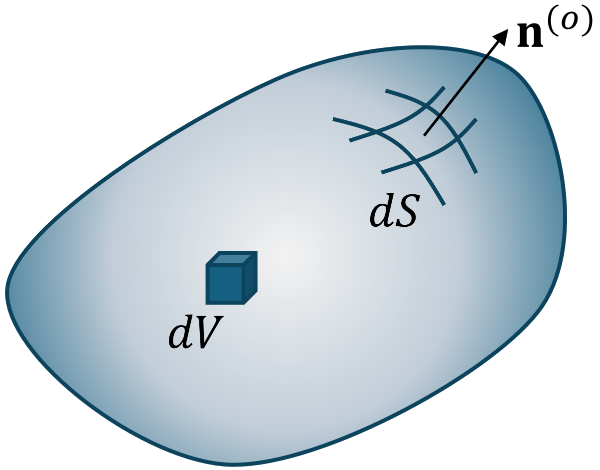
Summation subscript notation

- Divergence: $\nabla.\mathbf{v} = \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3}$ is written as $\frac{\partial v_j}{\partial x_j}$

- Gradient: $\nabla\rho = \frac{\partial \rho}{\partial x_i}$

- Divergence Theorem

$$\int_S \mathbf{a}.\mathbf{n}^{(o)} dS = \int_V \nabla.\mathbf{a} dV$$



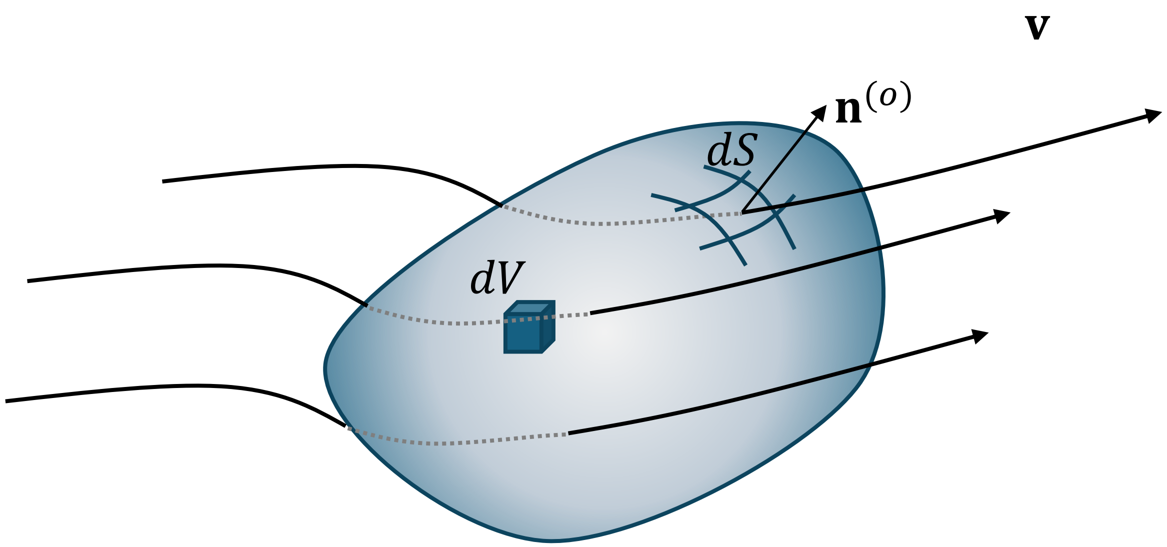
- Force exerted by the viscous stress tensor σ_{ij} on an

area $\mathbf{n}^{(o)}dS$ = $\sigma_{ij} n_j^{(o)} dS$

Equation of Continuity

Rate of change of mass in V $= -$ Net outflow of mass across S

Conservation form



Fixed control volume V with surface S

$$\frac{\partial}{\partial t}\int_V \rho dV = -\int_S \rho \mathbf{v}.\mathbf{n}^{(o)} dS$$

$$= -\int_V \nabla.(\rho\mathbf{v})\, dV$$

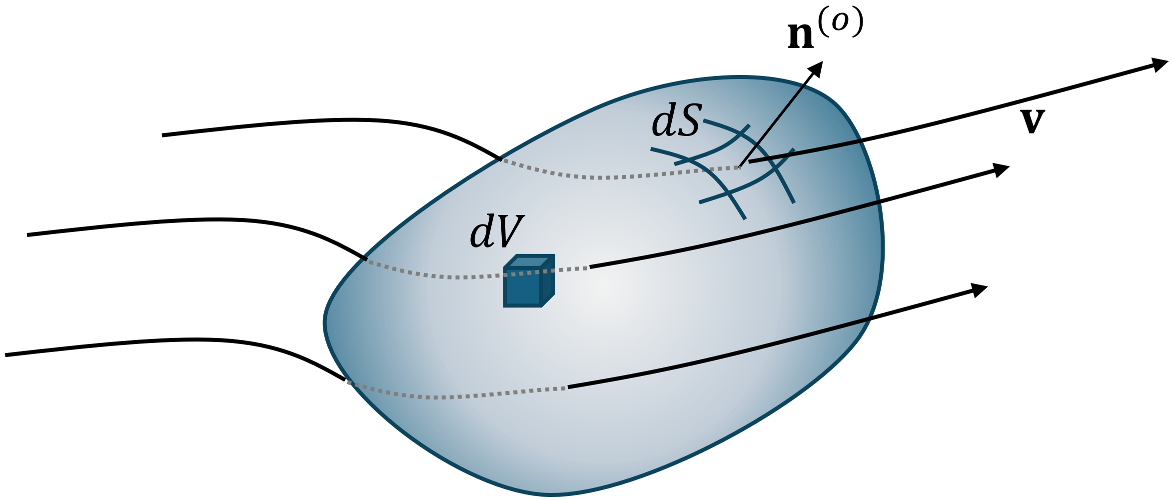
$$\frac{\partial \rho}{\partial t} + \nabla.(\rho\mathbf{v}) = 0$$

or

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho v_i}{\partial x_i} = 0$$

Equation of Momentum

$p_{ij} = p\delta_{ij} - \sigma_{ij}$



Fixed control volume V with surface S

Rate of change of momentum in V = Net force acting on S − Net outflow of momentum across S

$$\int_V \frac{\partial \rho v_i}{\partial t} dV = \int_S -p_{ij} n_j^{(o)} - v_i \rho v_j n_j^{(o)} dS$$

$$\int_V \frac{\partial \rho v_i}{\partial t} + \frac{\partial (p_{ij} + \rho v_i v_j)}{\partial x_j} dV = 0$$

so $$\frac{\partial \rho v_i}{\partial t} + \frac{\partial (p_{ij} + \rho v_i v_j)}{\partial x_j} = 0$$

Conservation form

Sound Waves

- Sound waves are small perturbations in pressure and density (p', ρ') from ambient conditions (p_o, ρ_o) we have,
- The changes are isentropic (a reversable change with no loss of heat) and proportional to each other, so

$$p' = \rho' \underbrace{\left(\frac{\partial p}{\partial \rho}\right)_s}_{\text{Bulk modulus of fluid}} = \rho' c_o^2 \qquad c_o^2 = \gamma p_o / \rho_o$$

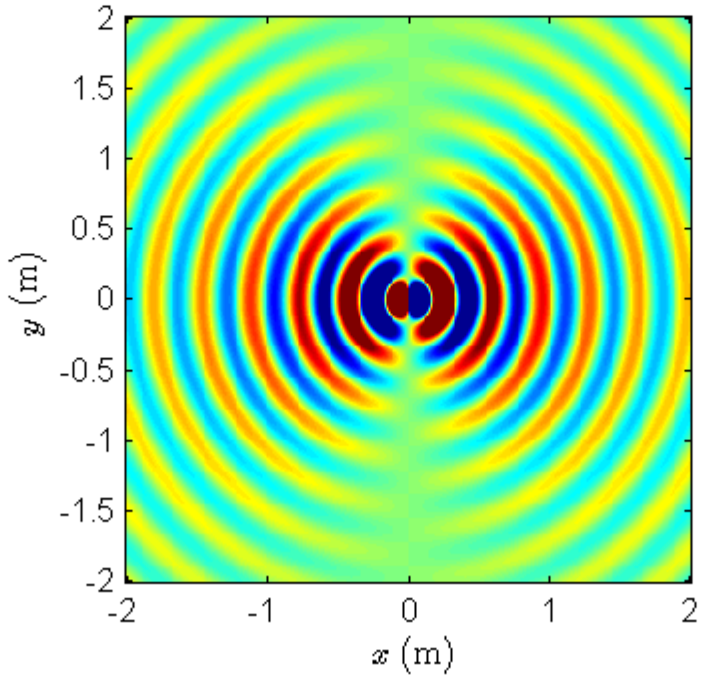
Summary

- Continuity: $$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho v_i)}{\partial x_i} = 0$$

- Momentum: $$\frac{\partial \rho v_i}{\partial t} + \frac{\partial(p_{ij}+\rho v_i v_j)}{\partial x_j} = 0$$

- Isentropic conditions: $p' = \rho' c_o^2$

Linear Acoustics Assumptions



- Linear acoustics is the study of acoustics without flow where SPLs are less than about 140dB (re 20 μPa) so

$p = p_o + p'$ where $p' \ll p_o$

$\rho = \rho_o + \rho'$ where $\rho' \ll \rho_o$

- The sound waves are isentropic
- Viscous effects can be ignored
- Velocity perturbation are small compared to the speed of sound

Acoustic wave equation

Obtained by applying the continuity and momentum equations to very small disturbances in a stationary atmosphere

Continuity

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho v_i)}{\partial x_i} = 0 \qquad \rho = \rho_o + \rho'$$

$$\frac{\partial \rho'}{\partial t} + \frac{\partial (\rho_o v_i + \rho' v_i)}{\partial x_i} = 0 \qquad \rho' \ll \rho_o$$

$$\frac{\partial \rho'}{\partial t} + \rho_o \frac{\partial v_i}{\partial x_i} = 0$$

Acoustic wave equation

Momentum

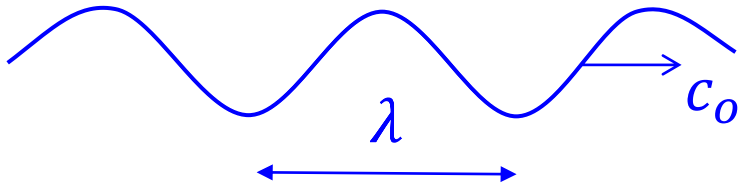
$$\frac{\partial(\rho v_i)}{\partial t}+\frac{\partial(p_{ij}+\rho v_i v_j)}{\partial x_j}=0$$

$$p_{ij}=p\delta_{ij}=(p_o+p')\delta_{ij}$$

$$\rho' \ll \rho_o$$

$$\rho_o\frac{\partial v_i}{\partial t}+\frac{\partial(p_o+p'+\rho_o v_i v_j)}{\partial x_i}=0$$

$$\rho_o\frac{\partial v_i}{\partial t}+\rho_o\frac{\partial v_i v_j}{\partial x_i}+\frac{\partial p'}{\partial x_i}=0$$



Timescale $T \sim \lambda/c_o$

$$\frac{\partial v_i}{\partial t}\sim\frac{v_i}{T}\sim\frac{v_i}{\lambda}c_o$$

$$\frac{\partial v_i v_j}{\partial x_i}\sim\frac{v_i^2}{\lambda}$$

$$\frac{v_i}{\lambda}c_o >> \frac{v_i^2}{\lambda}$$

$$\rho_o\frac{\partial v_i}{\partial t}+\frac{\partial p'}{\partial x_i}=0$$

"Acoustic momentum equation"

Acoustic wave equation

$$\boxed{p' = \rho' c_o^2}$$

Continuity

$$\frac{\partial \rho'}{\partial t} + \rho_o \frac{\partial v_i}{\partial x_i} = 0 \xrightarrow{\partial/\partial t} \frac{\partial^2 \rho'}{\partial t^2} + \rho_o \frac{\partial^2 v_i}{\partial x_i \partial t} = 0 \qquad \frac{1}{c_o^2}\frac{\partial^2 p'}{\partial t^2} + \rho_o \frac{\partial^2 v_i}{\partial x_i \partial t} = 0$$

Momentum

$$\rho_o \frac{\partial v_i}{\partial t} + \frac{\partial p'}{\partial x_i} = 0 \xrightarrow{\partial/\partial x_i} \rho_o \frac{\partial^2 v_i}{\partial x_i \partial t} + \frac{\partial^2 p'}{\partial x_i^2} = 0$$

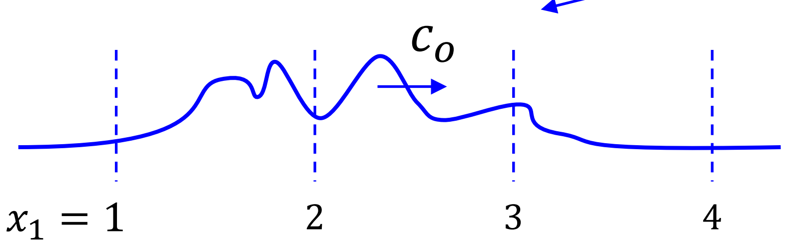
$$\text{Subtract} \longrightarrow \frac{1}{c_o^2}\frac{\partial^2 p'}{\partial t^2} - \frac{\partial^2 p'}{\partial x_i^2} = 0$$

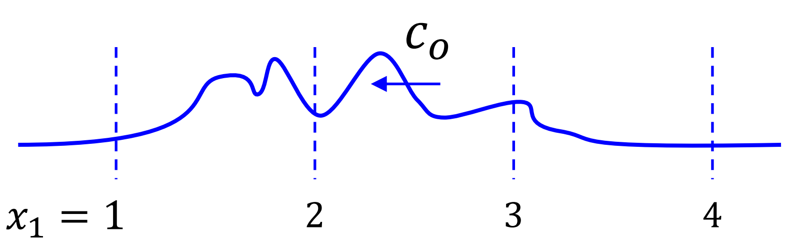
Plane wave solution

$$\frac{1}{c_o^2}\frac{\partial^2 p'}{\partial t^2} - \frac{\partial^2 p'}{\partial x_1^2} = 0$$

which has the general solution

$$p'(x_1, t) = f\left(t - \frac{x_1}{c_o}\right) + g\left(t + \frac{x_1}{c_o}\right)$$





Spherical wave solution

$$\frac{1}{c_o^2}\frac{\partial^2 p'}{\partial t^2} - \nabla^2 p' = 0$$

$$\nabla^2 p' = \frac{1}{r}\frac{\partial^2 r p'}{\partial r} + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial p'}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 p'}{\partial\phi}$$

$$\frac{1}{c_o^2}\frac{\partial^2 p'}{\partial t^2} - \frac{1}{r}\frac{\partial^2 r p'}{\partial r^2} = 0$$

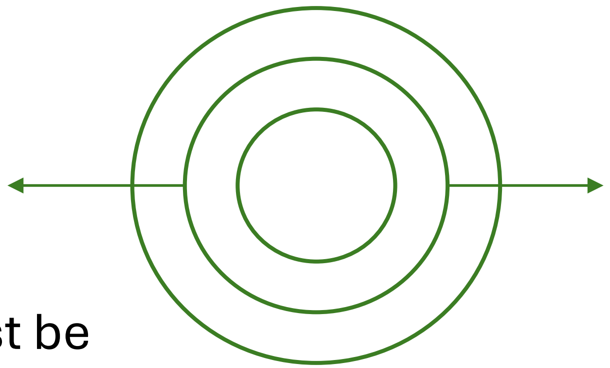
Multiplying through by r

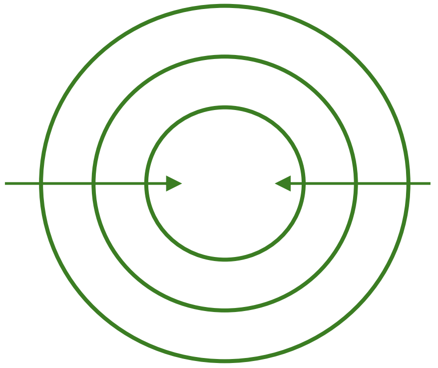
$$\frac{1}{c_o^2}\frac{\partial^2 r p'}{\partial t^2} - \frac{\partial^2 r p'}{\partial r^2} = 0$$

gives the 1D wave equation for rp'

Spherical wave solution

$$rp'(r,t) = f(t - r/c_o) + g(t + r/c_o)$$





so the general solution must be

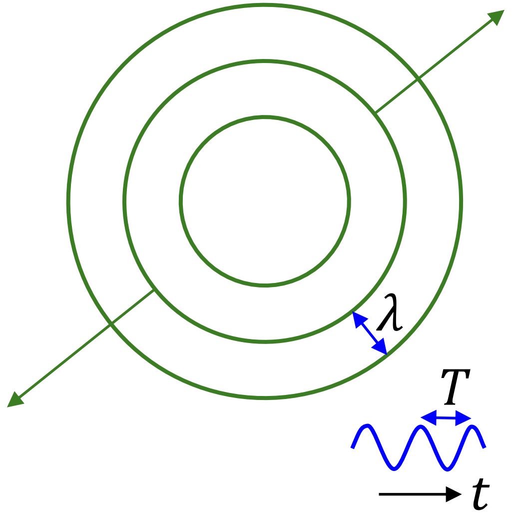
$$p'(r,t) = \frac{f(t - r/c_o)}{r}$$

Except for the rare case of incoming focused waves

We see that the pressure amplitude of an outgoing acoustic wave decays inversely with radius in 3D

Harmonic time dependence

$$p'(r,t) = \frac{f(t - r/c_o)}{r} = \frac{A\cos(\omega t - \omega r/c_o - \phi)}{r}$$



$$p'(r,t) = \frac{Re\{A\, e^{-i\omega t + ikr + i\phi}\}}{r}$$

$$\boldsymbol{k = \omega/c_o}$$

$$p'(r,t) = Re\{\hat{p}(r) e^{-i\omega t}\}$$

Now we introduce $\widehat{A} = Ae^{i\phi}$

$$\hat{p}(r) = \frac{\hat{A}e^{ikr}}{r}$$

Acoustic wave number k

The wave repeats each wavelength

So

$$\boldsymbol{k\lambda = \frac{\omega\lambda}{c_o} = 2\pi}$$

$$\boldsymbol{k = 2\pi/\lambda}$$

Harmonic time dependence

Since $p'(x_i, t) = Re\{\hat{p}(x_i)e^{-i\omega t}\}$ $v_i(x_i, t) = Re\{\widehat{v_i}e^{-i\omega t}\}$

The acoustic momentum equation $$\rho_o \frac{\partial v_i}{\partial t} + \frac{\partial p'}{\partial x_i} = 0$$

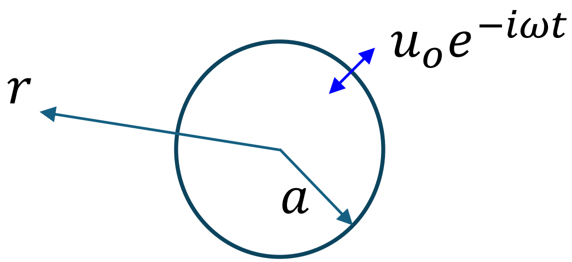
$$i\omega\rho_o\widehat{v_i} = \frac{\partial \hat{p}}{\partial x_i}$$

or

$$i\omega\rho_o\hat{\mathbf{v}} = \nabla\hat{p}$$

Sound generated by a small sphere

$$\boxed{\hat{p}(r) = \frac{\hat{A}e^{ikr}}{r}}$$



Acoustic momentum equation

$$\boxed{i\omega\rho_o \hat{v}_i = \frac{\partial \hat{p}}{\partial x_i}}$$

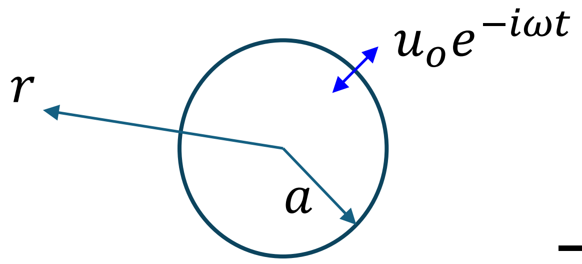
at the sphere $\hat{v}_r = u_o$

$$i\omega\rho_o \hat{v}_r = \frac{\partial \hat{p}}{\partial r}$$

$$\frac{1}{i\omega\rho_o}\left[\frac{\partial \hat{p}}{\partial r}\right]_{r=a} = u_o$$

Sound generated by a small sphere

$$\frac{\partial \hat{p}}{\partial r} = ik\frac{\hat{A}e^{ikr}}{r} - \frac{\hat{A}e^{ikr}}{r^2}$$

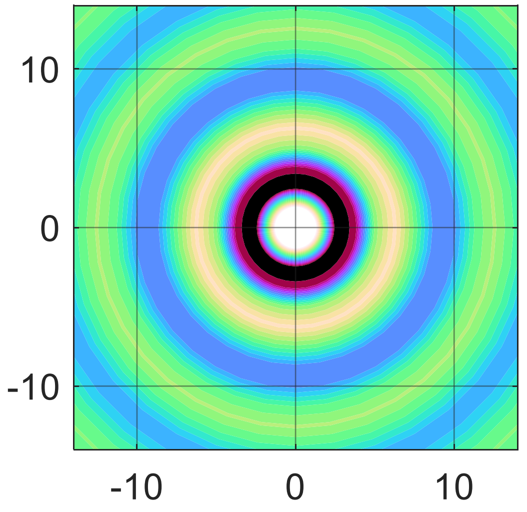


$$\frac{1}{i\omega\rho_o}\left[\frac{\partial \hat{p}}{\partial r}\right]_{r=a} = u_o$$

Solving for the unknown constant *A*

gives $$\hat{A} = -\frac{i\omega\rho_o a^2 u_o e^{-ika}}{1 - ika}$$

$$\hat{p} = -\frac{i\omega\rho_o a^2 u_o e^{ik(r-a)}}{(1 - ika)r}$$



Superposition and the far field

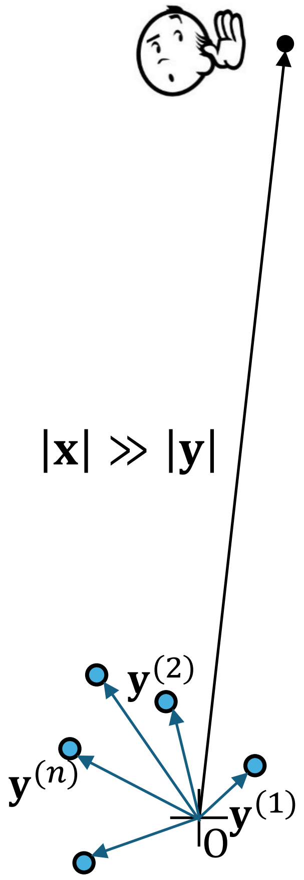
Monopole source

$$\hat{p}(r) = \frac{\hat{A}e^{ikr}}{r}$$

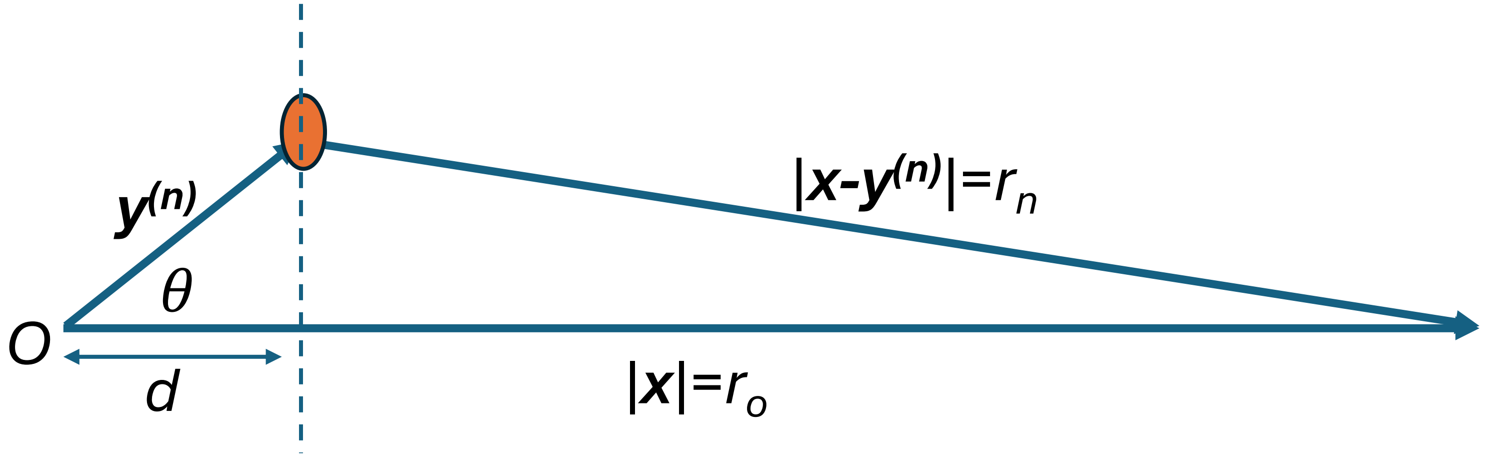
Consider a local region containing many monopoles radiating to the far field. Sound from the sources just combines linearly, so

$$\hat{p}(\mathbf{x}) = \sum_{n=1}^{N} \frac{\widehat{A_n} e^{ik\left|\mathbf{x}-\mathbf{y}^{(n)}\right|}}{\left|\mathbf{x}-\mathbf{y}^{(n)}\right|}$$

$$r_n = |\boldsymbol{x} - \boldsymbol{y}^{(n)}|$$



Far Field Approximation



$$d = \left|\boldsymbol{y}^{(\boldsymbol{n})}\right| cos\theta = \boldsymbol{y}^{(\boldsymbol{n})}.\boldsymbol{x}/|\boldsymbol{x}|$$

$$\begin{aligned}\left|\boldsymbol{x} - \boldsymbol{y}^{(\boldsymbol{n})}\right| &\approx |\boldsymbol{x}| - d \\ &= |\boldsymbol{x}| - \boldsymbol{y}^{(\boldsymbol{n})}.\boldsymbol{x}/|\boldsymbol{x}|\end{aligned}$$

Superposition and the far field

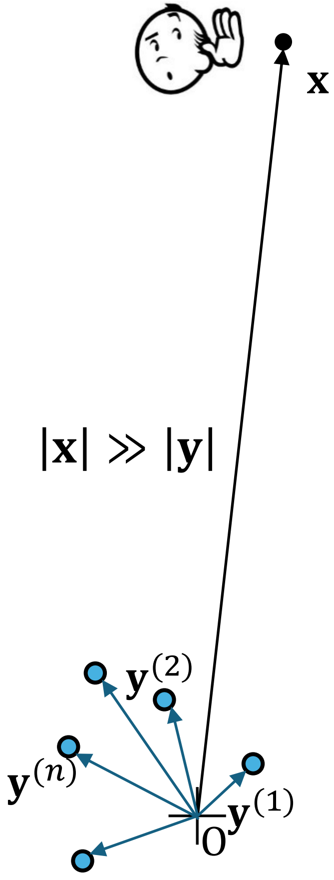
So in the far field

$$\hat{p}(\mathbf{x}) \approx \sum_{n=1}^{N} \frac{\widehat{A_n} e^{ik(|\mathbf{x}| - \mathbf{y}^{(n)}.\mathbf{x}/|\mathbf{x}|)}}{(|\mathbf{x}| - \mathbf{y}^{(n)}.\mathbf{x}/|\mathbf{x}|)}$$

or

$$\hat{p}(\mathbf{x}) \approx \frac{e^{ik|\mathbf{x}|}}{|\mathbf{x}|} \sum_{n=1}^{N} \widehat{A_n} e^{-ik\mathbf{y}^{(n)}.\mathbf{x}/|\mathbf{x}|}$$

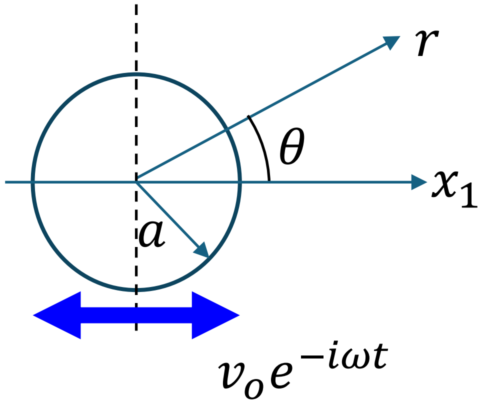
Controls retarded time regardless of $|\boldsymbol{x}|$ *and* $|\mathbf{y}|$ because phase shift is relative to source position



Oscillating sphere

Boundary condition

$[\hat{v}_r]_{r=a} = v_o \cos\theta$



Acoustic momentum equation

$$\boxed{i\omega\rho_o \hat{v}_r = \frac{\partial \hat{p}}{\partial r}}$$

Needs a Cosine dependence

Ffowcs William's Observation

Derivatives of solutions to the wave equation are also solutions to the wave equation

If p_M is a solution to the wave equation then so is $\frac{\partial^n p_M}{\partial x_j^n}$

$$\frac{1}{c_o^2}\frac{\partial^2}{\partial t^2}\left(\frac{\partial^n p_M}{\partial x_j^n}\right) - \frac{\partial^2}{\partial x_i^2}\left(\frac{\partial^n p_M}{\partial x_j^n}\right)$$

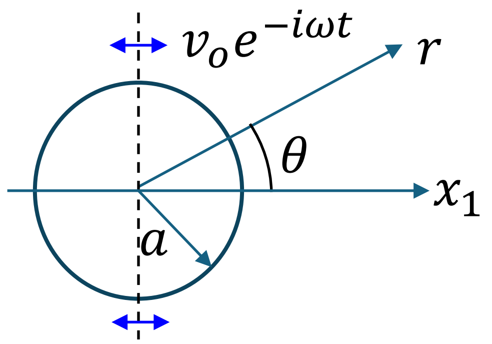
$$= \frac{\partial^n}{\partial x_j^n}\left(\frac{1}{c_o^2}\frac{\partial^2 p_M}{\partial t^2} - \frac{\partial^2 p_M}{\partial x_i^2}\right) = 0$$

Oscillating sphere

By differentiating the spherical solution with respect to x_1

$$\hat{p}(r) = \frac{\partial}{\partial x_1}\left(\frac{\hat{A}e^{ikr}}{r}\right) = ik\cos\theta\left(\frac{\hat{A}e^{ikr}}{r}\right)\left(1 - \frac{1}{ikr}\right)$$

$i\omega\rho_o v_o cos\theta = \frac{\partial\hat{p}}{\partial r}$. Is evaluated at $r = a$ to match to the b. c. to give $\hat{A}$



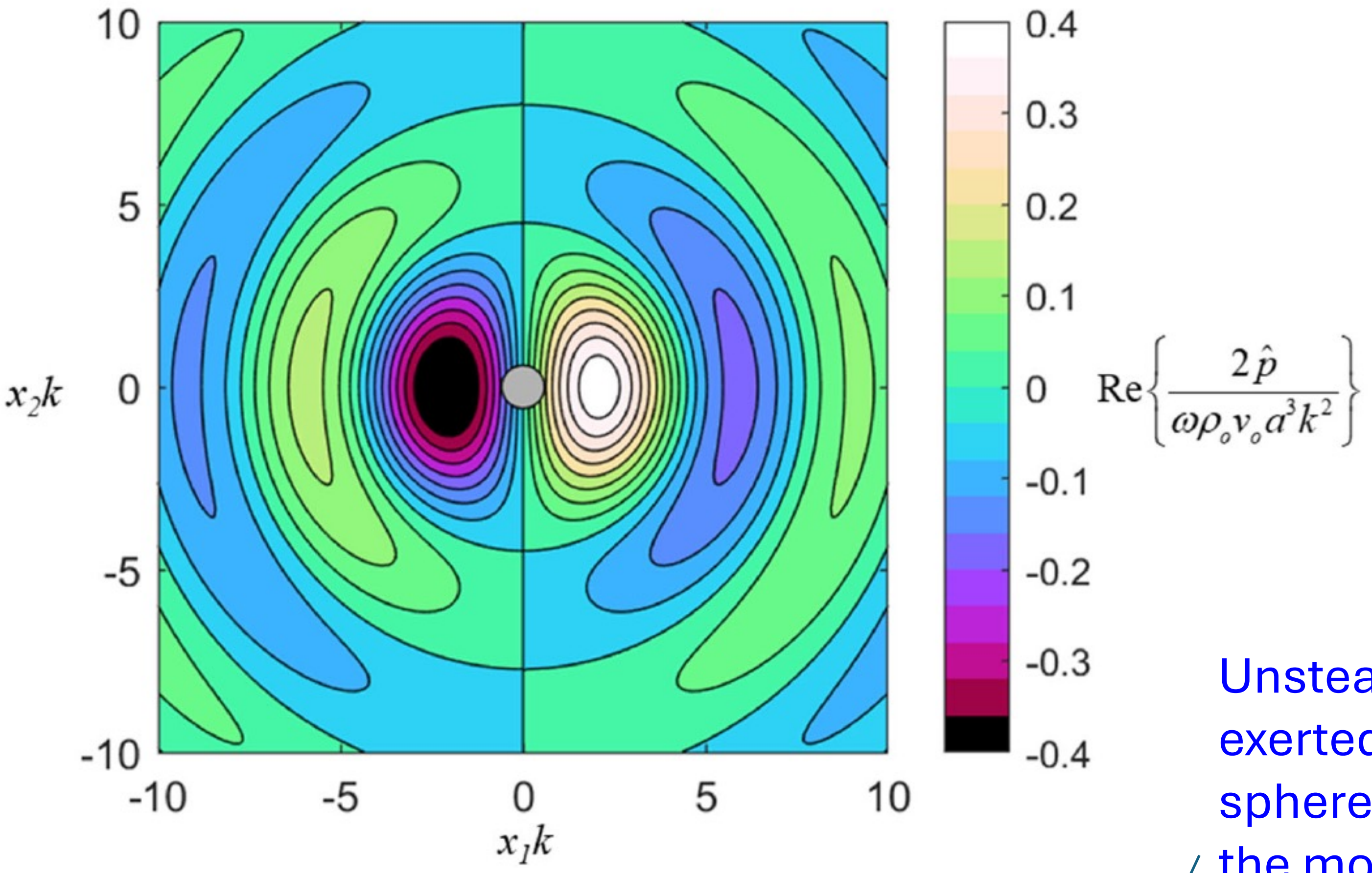
Boundary condition

$[\hat{v}_r]_{r=a} = v_o \cos\theta$

$$\hat{p}(r) = ik\cos\theta\left(\frac{i\omega\rho_o v_o a^3 e^{ikr}}{2r}\right)\left(1 - \frac{1}{ikr}\right)$$ assuming $ka \ll 1$

The $\cos\theta$ dependence means that the sound is beamed along x_1 and has opposite phase at $\pm x_1$. This fundamental wave field is called a **dipole**

Wave field

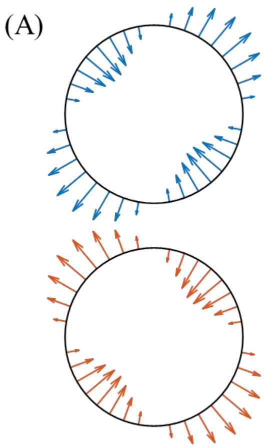


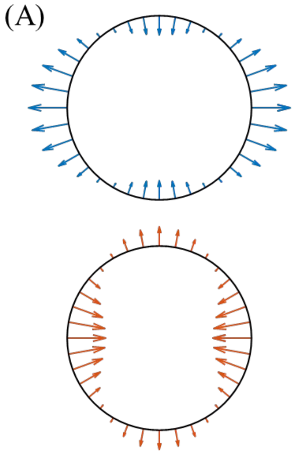
2 lobes

Unsteady force exerted on the sphere to achieve the motion

$$\hat{p}(r) = ik\cos\theta\left(\frac{i\omega\rho_o v_o a^3 e^{ikr}}{2r}\right)\left(1-\frac{1}{ikr}\right) = -ik\cos\theta\, e^{ikr}\left(\frac{\hat{F}}{4\pi r}\right)\left(1-\frac{1}{ikr}\right)$$

Swishing Motions





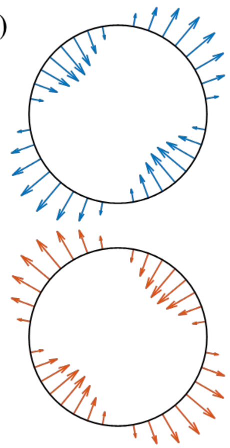
Lateral Motion

$$[\hat{v}_r]_{r=a} = v_o \cos\theta sin\theta cos\phi$$

Longitudinal Motion

$$[\hat{v}_r]_{r=a} = v_o\,(1 - 3\cos^2\theta)$$

Swishing Motions(Latreral)



$$\hat{p}(r) = \frac{\partial^2}{\partial x_1 \partial x_2}\left(\frac{\hat{A}e^{ikr}}{r}\right) = \frac{x_1 x_2}{r^2}\left(\frac{\hat{A}e^{ikr}}{r}\right)\left(\frac{3}{r^2} - \frac{3ik}{r} - k^2\right)$$

$$\frac{x_1 x_2}{r^2} = \cos\theta sin\theta cos\phi$$

Use the acoustic momentum equation $i\omega\rho_o \hat{v}_r = \frac{\partial \hat{p}}{\partial r}$, and evaluate this at $r = a$ to match to the b. c. to give $\hat{A}$ and the sound field as

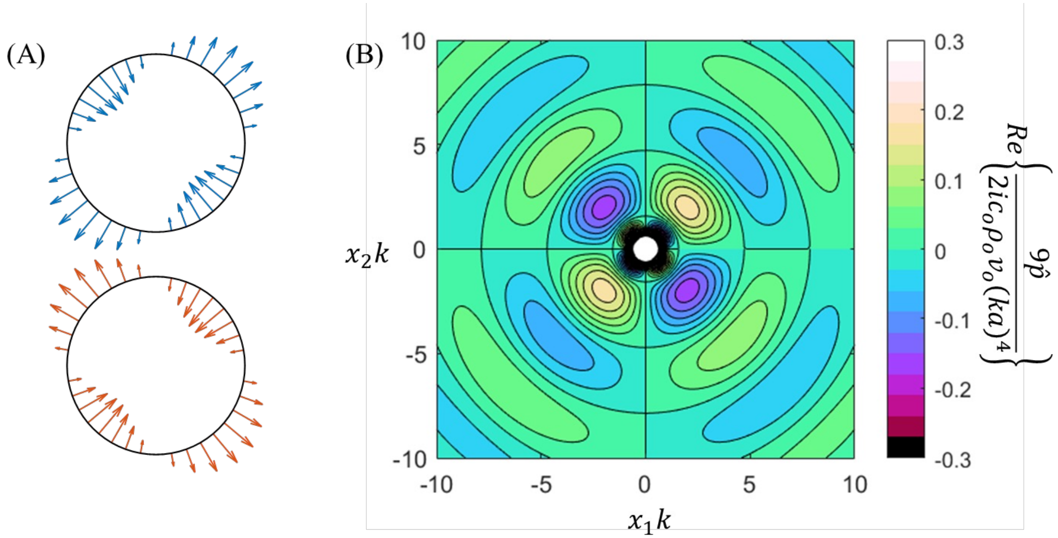
Lateral Motion

Assuming $ka \ll 1$ and $kr \gg 1$

$$[\hat{v}_r]_{r=a} = v_o \cos\theta sin\theta cos\phi$$

$$\hat{p}(r) = \frac{2(ka)^2}{9}\left(\frac{x_1 x_2}{r^2}\right)\left(\frac{i\omega\rho_o v_o a^2 e^{ikr}}{r}\right)$$

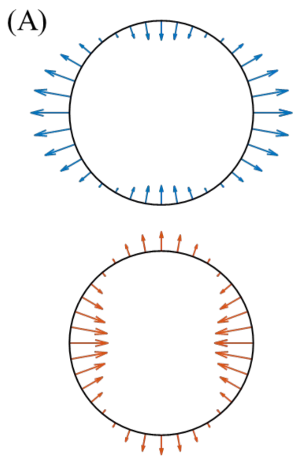
Swishing Motions (Lateral)



$$\hat{p}(r) = \frac{2(ka)^2}{9}\left(\frac{x_1 x_2}{r^2}\right)\left(\frac{i\omega\rho_o v_o a^2 e^{ikr}}{r}\right)$$

Swishing Motions(Longitudinal)

$$\hat{p}(r) = \frac{\partial^2}{\partial x_1^2}\left(\frac{\hat{A}e^{ikr}}{r}\right) = \left(\frac{\hat{A}e^{ikr}}{r}\right)\left(\frac{ik}{r} - \frac{1}{r^2}\right) - \frac{3x_1^2}{r^2}\left(\frac{\hat{A}e^{ikr}}{r}\right)\left(\frac{ik}{r} - \frac{1}{r^2} + \frac{k^2}{3}\right)$$



$$\frac{x_1^2}{r^2} = \cos^2\theta \qquad \hat{p}(r) = B(r)(1 - 3\cos^2\theta) - k^2\cos^2\theta\left(\frac{\hat{A}e^{ikr}}{r}\right)$$

This is not an exact match to the surface motion because of the k^2 term

However if we drop terms of order $(ka)^2$ *in the near field (incompressible assumption)* to solve for *A* then in the far field *kr>>1*

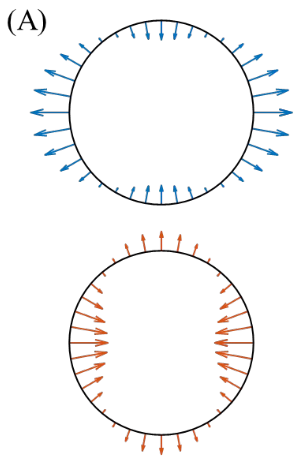
Longitudinal Motion

$$[\hat{v}_r]_{r=a} = v_o\,(1 - 3\cos^2\theta)$$

$$\hat{p}(r) = \frac{(ka)^2}{6}\left(\frac{x_1^2}{r^2}\right)\left(\frac{i\omega\rho_o v_o a^2 e^{ikr}}{r}\right)$$

But this is the wrong result

Swishing Motions(Longitudinal Corrected)



$$\hat{p}(r) = \frac{\partial^2}{\partial x_1^2}\left(\frac{\hat{A}e^{ikr}}{r}\right) - k^2\left(\frac{\hat{A}e^{ikr}}{r}\right) = \left(1 - \frac{3x_1^2}{r^2}\right)\left(\frac{\hat{A}e^{ikr}}{r}\right)\left(\frac{ik}{r} - \frac{1}{r^2} + \frac{k^2}{3}\right)$$

This is now an exact match to the surface motion

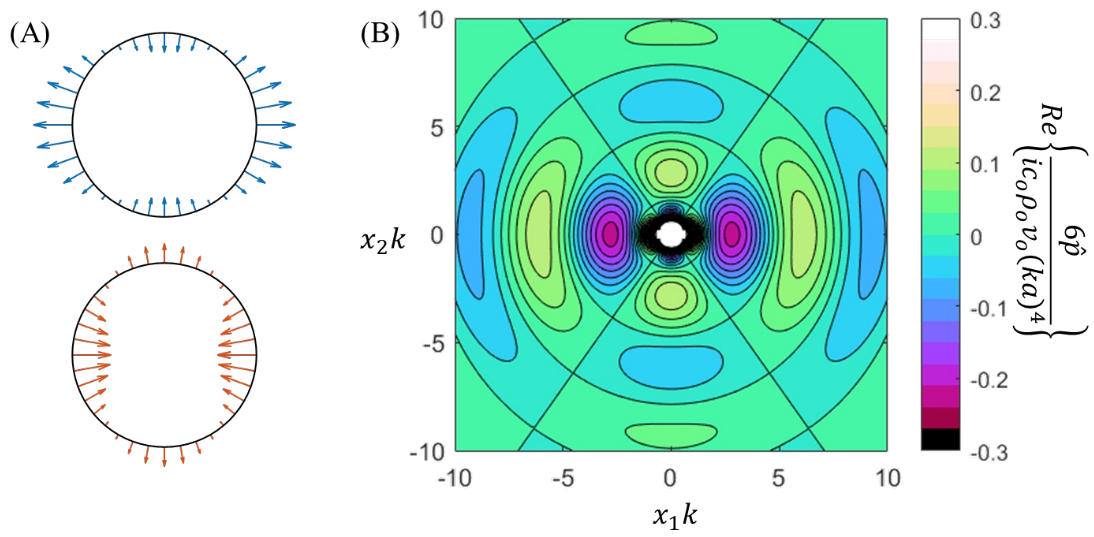
$$\hat{p}(r) = \frac{(ka)^2}{18}\left(\frac{3x_1^2}{r^2} - \mathbf{1}\right)\left(\frac{i\omega\rho_o v_o a^2 e^{ikr}}{r}\right)$$

Longitudinal Motion

$$[\hat{v}_r]_{r=a} = v_o\,(1 - 3\cos^2\theta)$$

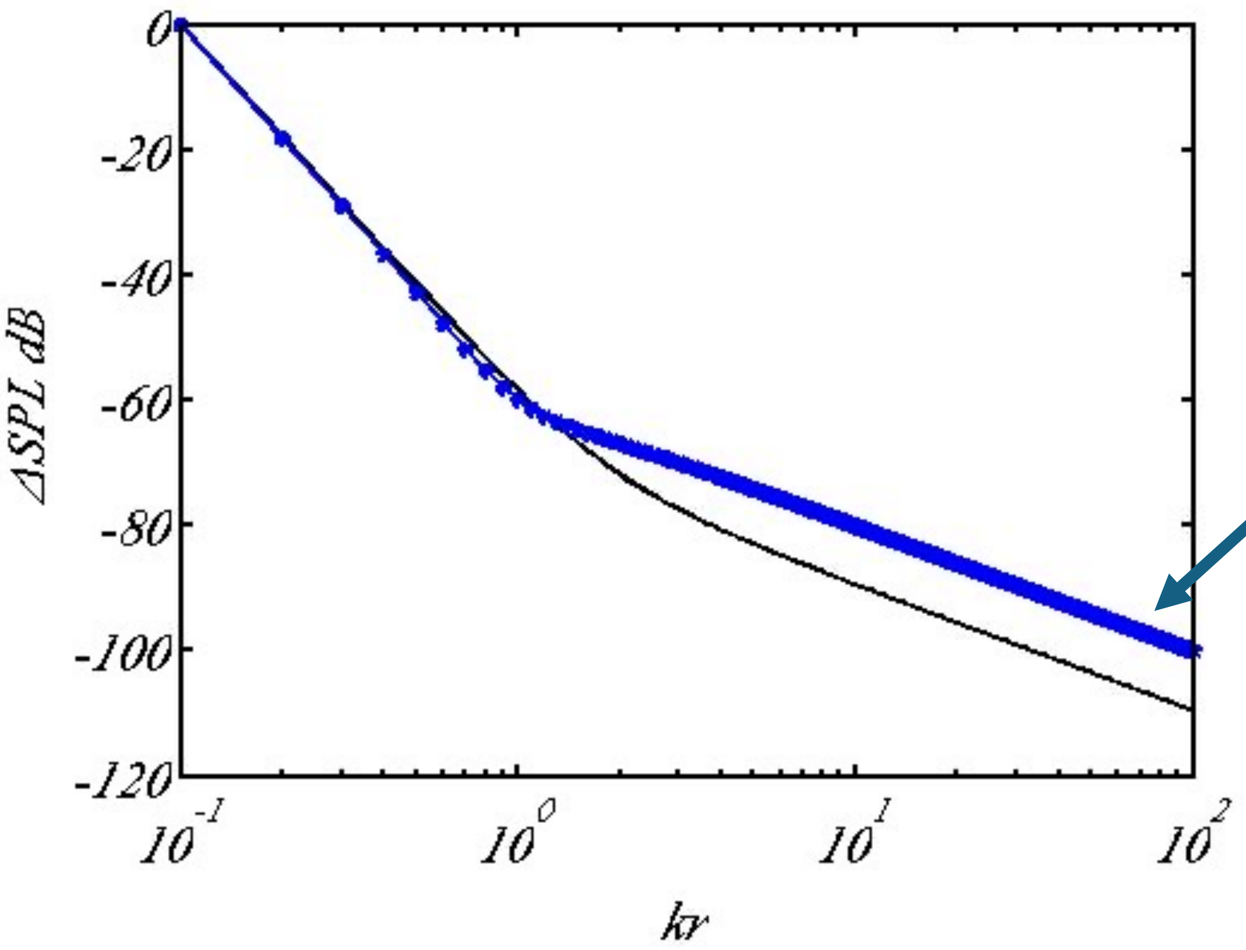
This is the correct result>>>>we needed to include a term of order *k*²

Swishing Motions



$$\hat{p}(r) = \frac{(ka)^2}{6}\left(\frac{3x_1^2}{r^2} - 1\right)\left(\frac{i\omega\rho_o v_o a^2 e^{ikr}}{r}\right)\left(\frac{ik}{r} - \frac{1}{r^2} + \frac{k^2}{3}\right)$$

The Effect of the Approximation at 45 deg.

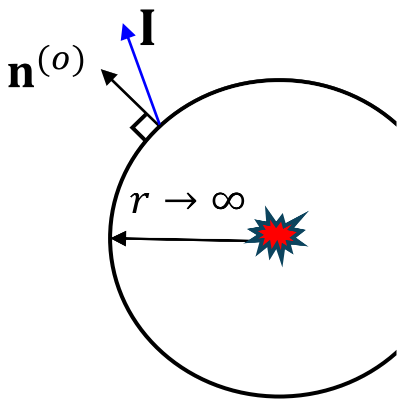


10 dB error
in acoustic far field

See Lighthill's Original paper on Aerodynamic Noise

C. Acoustic intensity, sound power

$$W_a = \int_S \mathbf{I}.\mathbf{n}^{(o)} dS$$
$$\mathbf{I} = E[p'\mathbf{v}]$$



$$\mathbf{I} = E[p'\mathbf{v}] = E\left[Re\{\hat{p}(\mathbf{x})e^{-i\omega t}\}Re\{\hat{v}_i(\mathbf{x})e^{-i\omega t}\}\right] = \cdots = \frac{1}{2}Re\{\hat{p}(\mathbf{x})\hat{v}_i^*(\mathbf{x})\}$$

In the far field of all of our sources $\hat{p}(\mathbf{x}) \sim e^{ikr}/r$ where $r = |\mathbf{x}|$, and the velocities associated with the sound waves will all be in the direction of r

So, the acoustic momentum equation becomes $i\omega\rho_o\hat{v}_r = \partial\hat{p}/\partial r$ which evaluates to $\hat{v}_r = \frac{\hat{p}}{i\omega\rho_o}\left(ik - \frac{1}{r}\right)$ and that the radial acoustic intensity component,

$$\underline{I_r = |\hat{p}|^2/2\rho_o c_o}$$

$$p'(\mathbf{x}, t) = Re\{\hat{p}(\mathbf{x})e^{-i\omega t}\}$$
$$v_i(\mathbf{x}, t) = Re\{\hat{v}_i(\mathbf{x})e^{-i\omega t}\}$$
$$i\omega\rho_o\hat{\mathbf{v}} = \nabla\hat{p}$$

Substituting our source pressure fields and integrating the power over a sphere of radius r centered on the source we obtain

$$W_a^{(mono)} = \frac{(\omega\rho_o Q)^2}{8\pi\rho_o c_o}$$

$$W_a^{(di)} = \frac{(kd)^2}{3}W_a^{(mono)}$$

$$W_a^{(long.quad)} = \frac{(kd)^4}{5}W_a^{(mono)}$$

$$W_a^{(lat.quad)} = \frac{(kd)^4}{15}W_a^{(mono)}$$