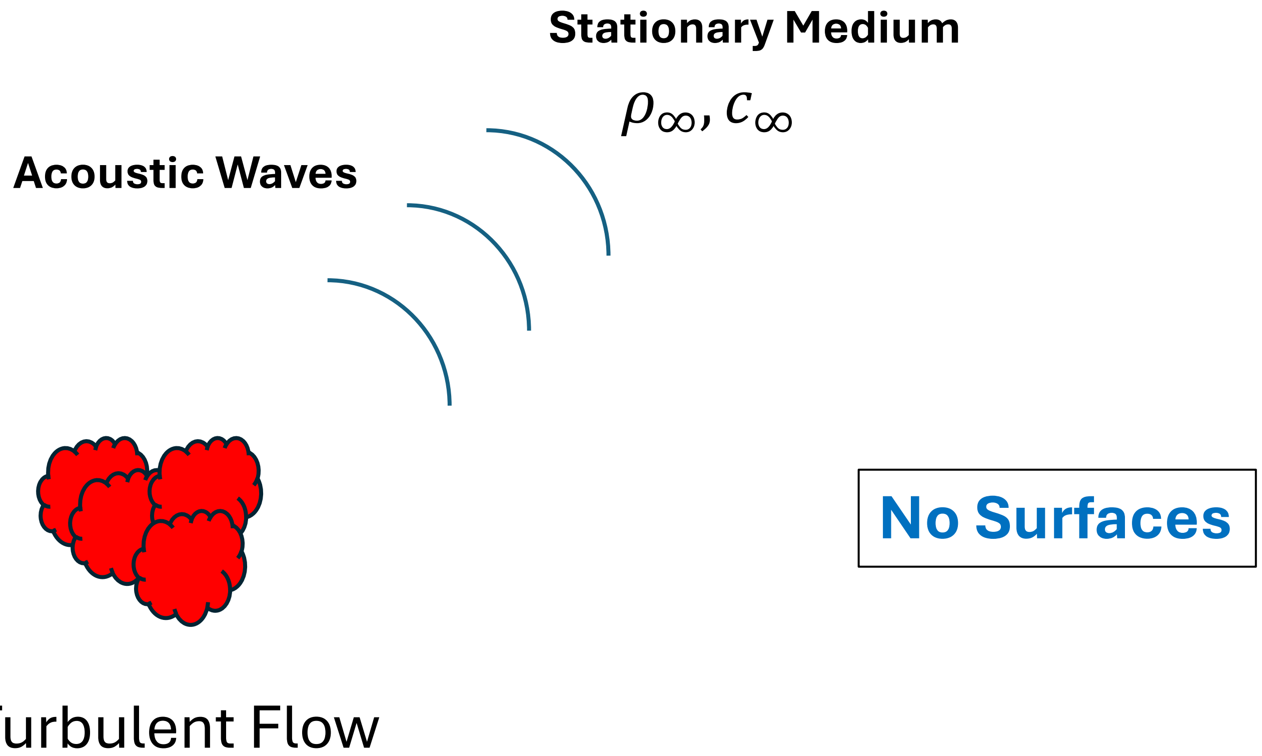
Lighthill’s Acoustic Analogy

Lighthill’s Acoustic Analogy Module

Outline of Topics

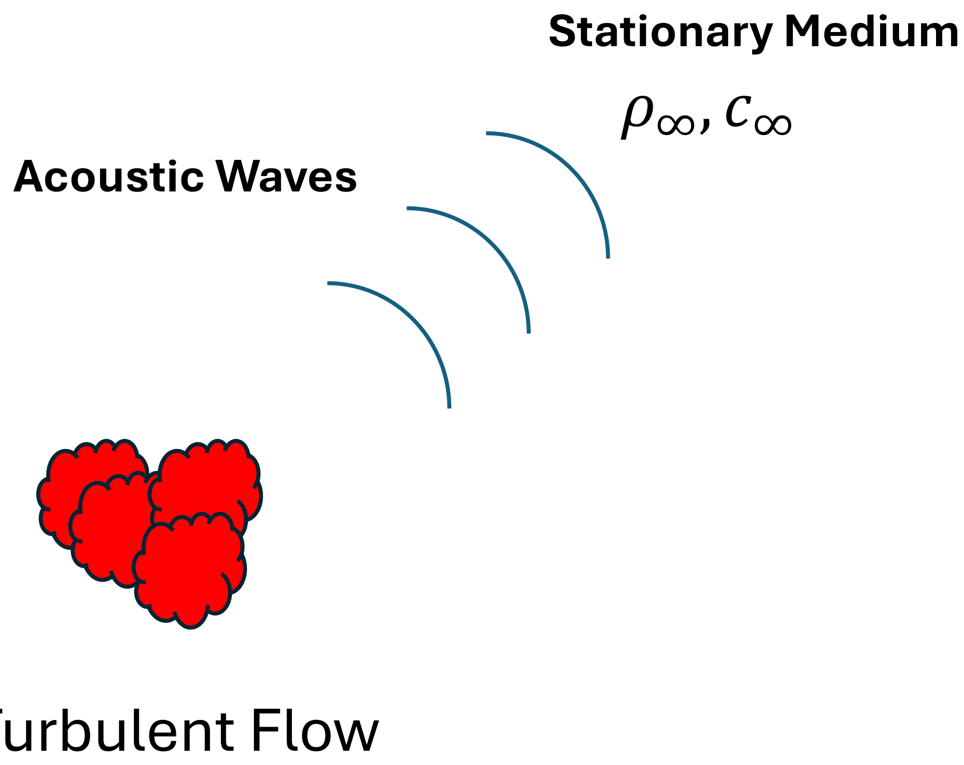
- **The Concept NS=0**
- **Lighthill’s Wave Equation**
- **The Free Field Solutions**
- **Limitations**
- **Near and Far Field Sound, U^8**
- **Surfaces, the FW-H equation**
- **Moving Sources**
- **Dipole, Thickness and Edge Noise**
- **Scaling on Flow Speed**

Lighthill’s Analogy



What is the sound level in the stationary medium?

Lighthill's Analogy



The Navier Stokes Equations Appy Everywhere

Momentum ***NS****=0*

Continuity $\dfrac{\partial M}{\partial t} = 0$

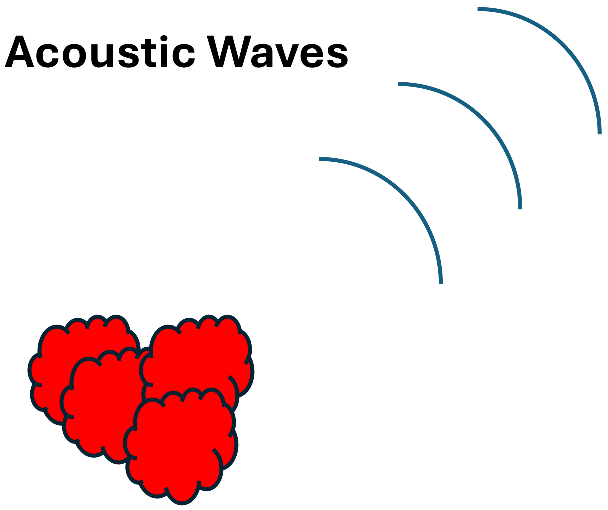
$$\nabla.\boldsymbol{NS} - \frac{\partial^2 M}{\partial t^2} = 0$$

Lighthill’s Analogy

Stationary Medium

ρ_∞, c_∞

Acoustic Waves



Turbulent Flow

In the Stationary medium

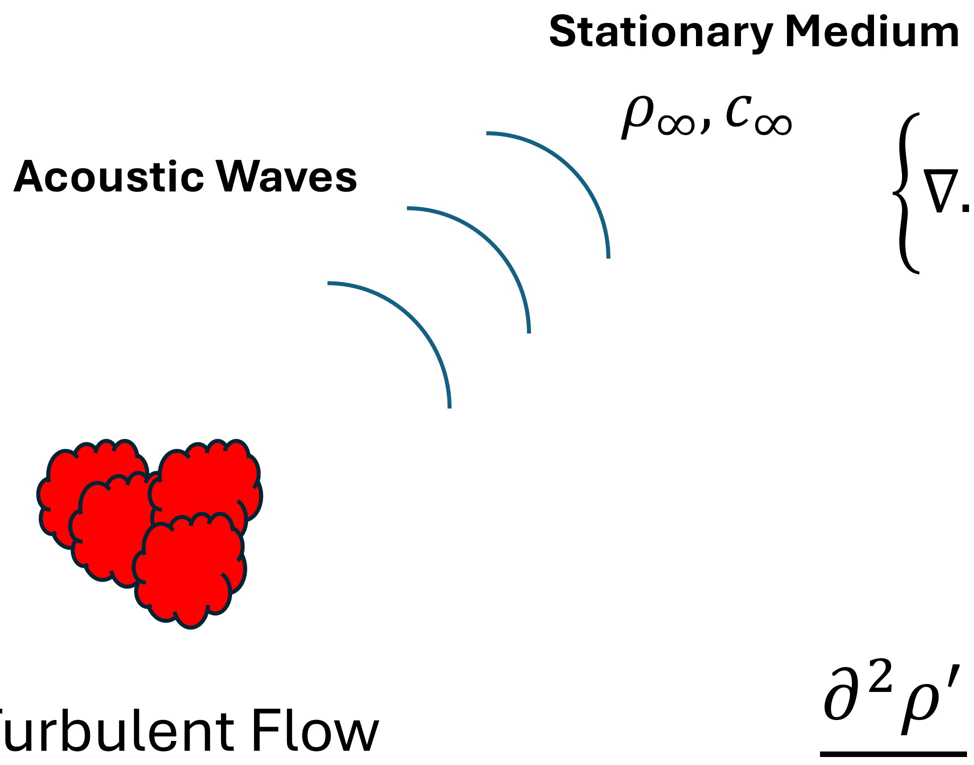
$$\frac{\partial^2 \rho'}{\partial t^2} - c_\infty^2 \nabla^2 \rho' = 0$$

Adding and subtracting this from

$$\nabla . \boldsymbol{NS} - \frac{\partial^2 M}{\partial t^2} = 0$$

$$\frac{\partial^2 \rho'}{\partial t^2} - c_\infty^2 \nabla^2 \rho' + \left\{ \nabla . \boldsymbol{NS} - \frac{\partial^2 M}{\partial t^2} - \left(\frac{\partial^2 \rho'}{\partial t^2} - c_\infty^2 \nabla^2 \rho' \right) \right\} = 0$$

Lighthill’s Analogy



Simplifying Terms

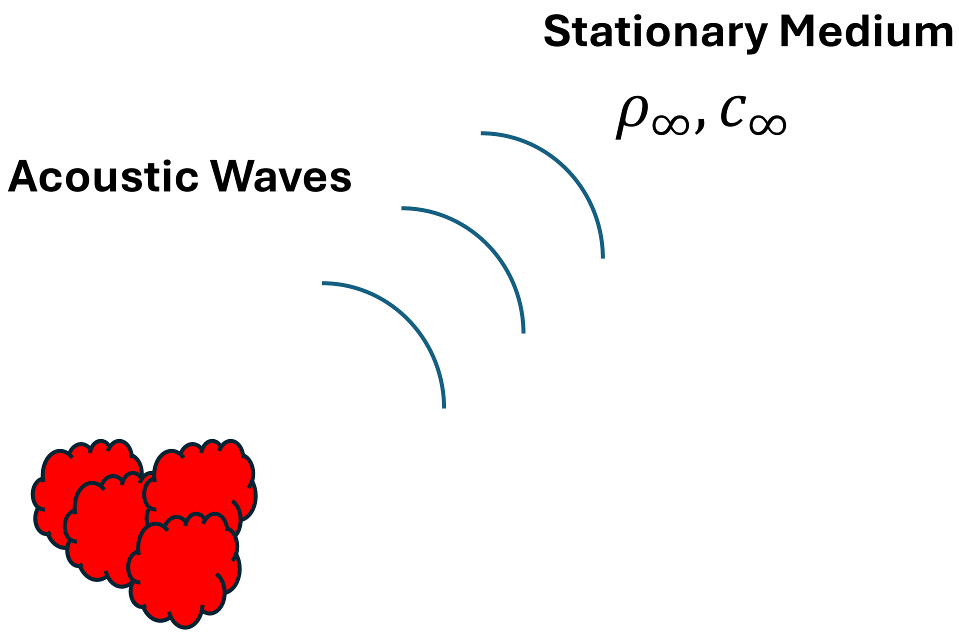
$$\left\{\nabla.\boldsymbol{NS} - \frac{\partial^2 M}{\partial t^2} - \left(\frac{\partial^2 \rho'}{\partial t^2} - c_\infty^2 \nabla^2 \rho'\right)\right\} = \frac{-\partial^2 T_{ij}}{\partial x_i \partial x_j}$$

$$T_{ij} = \rho v_i v_j + (p_{ij} - \rho' c_\infty^2 \delta_{ij})$$

$$\frac{\partial^2 \rho'}{\partial t^2} - c_\infty^2 \nabla^2 \rho' - \frac{\partial^2 T_{ij}}{\partial x_i \partial x_j} = 0$$

Zero in far field
stationary medium

Lighthill's Analogy



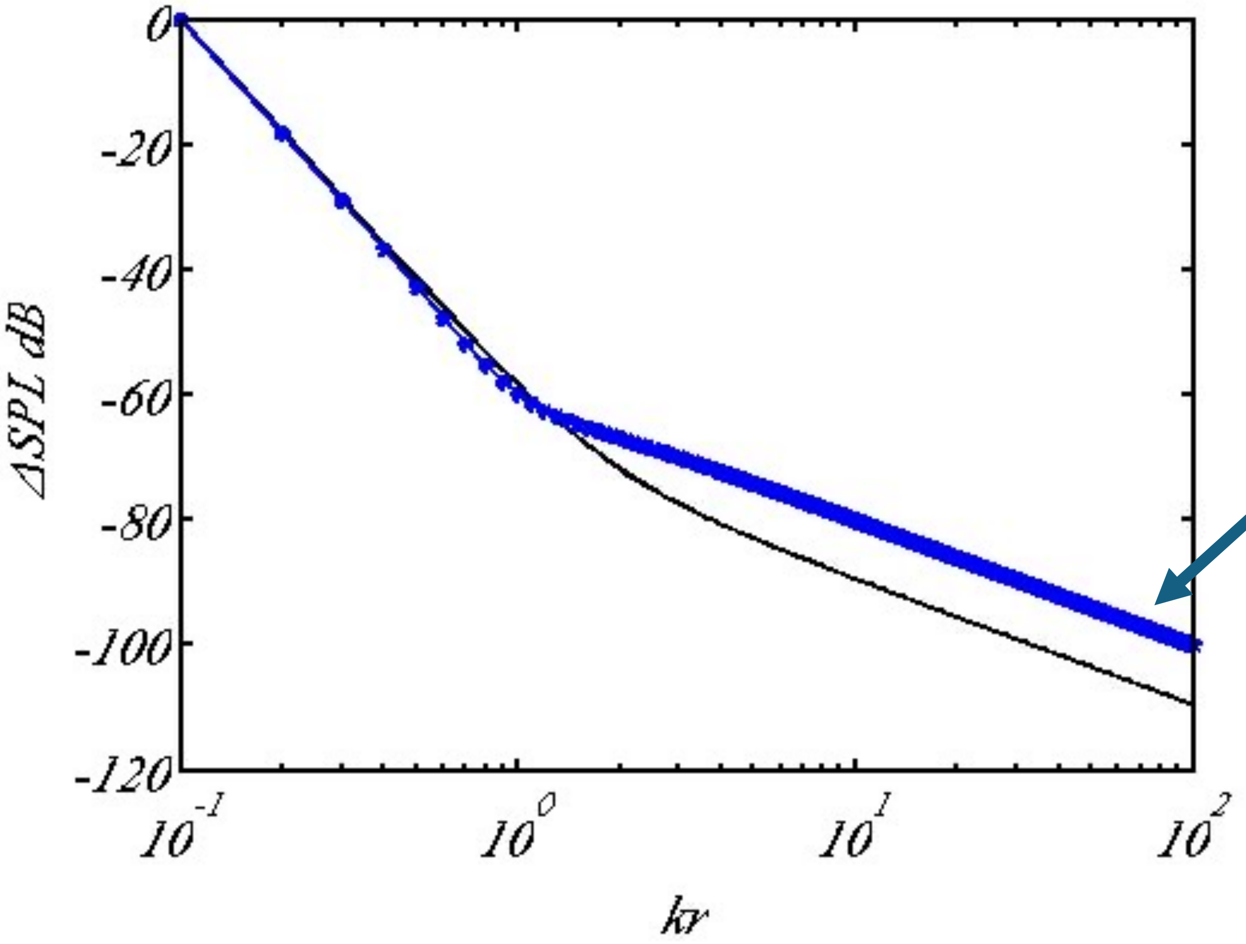
Turbulent Flow

Lighthill's equation is an exact re-arrangement of the equations of motion that asymptotes to the wave equation in the far field…it is an interpolation method!

$$\frac{\partial^2 \rho'}{\partial t^2} - c_\infty^2 \nabla^2 \rho' - \frac{\partial^2 T_{ij}}{\partial x_i \partial x_j} = 0$$

Tends to zero in far field stationary medium

The Effect of the incompressible source approximation at 45 deg. for longitudinal quadrupole



10 dB error in acoustic far field if near field incompressible approximation is used

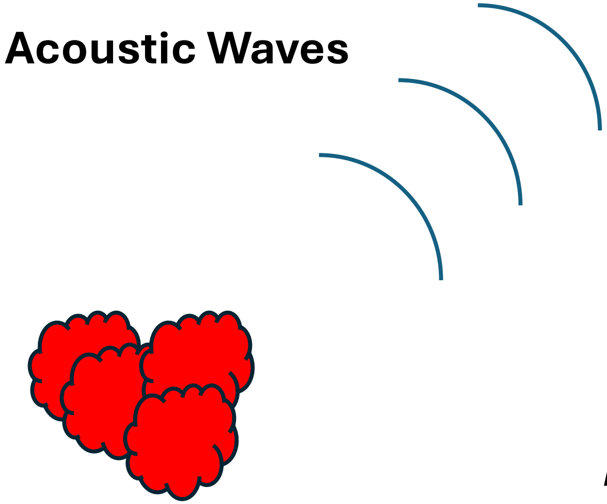
Lighthill's equation eliminates this type of error (in principle)

Lighthill’s Analogy

Stationary Medium

ρ_∞, c_∞

Acoustic Waves



Turbulent Flow

Lighthill’s Stress Tensor

$$T_{ij} = \rho v_i v_j + (p_{ij} - \rho' c_\infty^2 \delta_{ij})$$

$$\rho = \rho_\infty + \rho'. \qquad v_i = U_i + u_i$$

$$p_{ij} = p\delta_{ij} - \sigma_{ij}$$

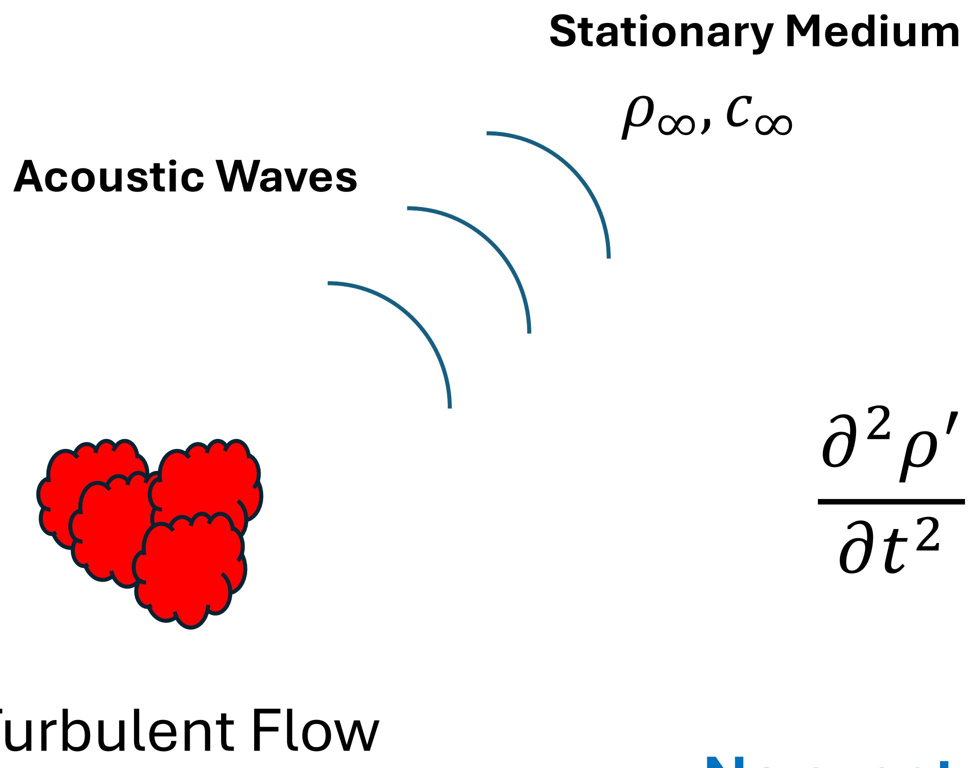
Mean shear Refraction

$$\rho v_i v_j = \rho_\infty U_i U_j + \rho' U_i U_j + \rho(U_i u_j + u_i U_j) + \rho_\infty u_i u_j + \rho' u_i u_j$$

Linearly coupled to pressure, so not independent of acoustic wave

Lighthill's Analogy

Lighthill's Stress Tensor



$$\frac{\partial^2 \rho'}{\partial t^2} - c_\infty^2 \nabla^2 \rho' - F(\rho') = \frac{\partial^2 T_{ij}}{\partial x_i \partial x_j} - F(\rho')$$

No exact analytical solutions to this equation

Greens Functions

$$\frac{\partial^2 \rho'}{\partial t^2} - c_\infty^2 \nabla^2 \rho' = \frac{\partial^2 T_{ij}}{\partial x_i \partial x_j}$$

Lighthill's equation is in the form of an inhomogeneous wave equation which can be solved using Greens functions

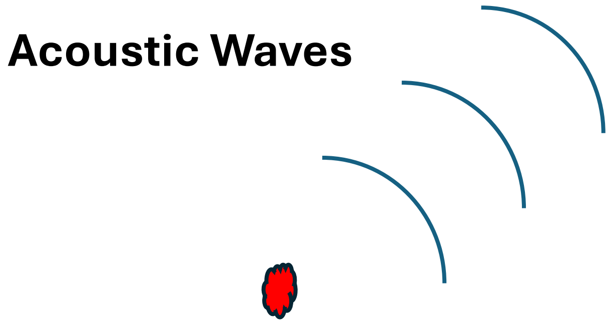
$$\frac{1}{c_\infty^2}\frac{\partial^2 \phi}{\partial \tau^2} - \nabla^2 \phi = Q(\boldsymbol{y}, \tau)$$

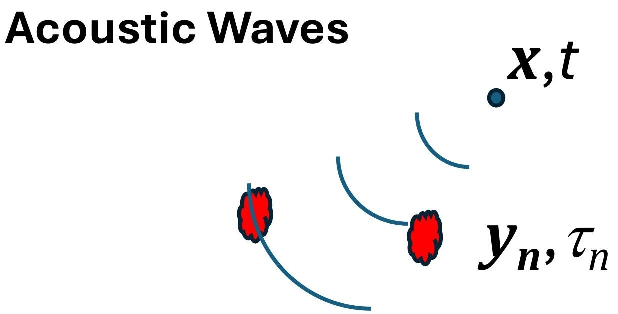
$$\phi(\boldsymbol{x}, t) = \int_V \int_{-T}^{T} Q(\boldsymbol{y}, \tau) G(\boldsymbol{x}, t | \boldsymbol{y}, \tau) dV d\tau$$

Greens Functions

$$\phi(\boldsymbol{x},t) = \int_V \int_{-T}^{T} Q(\boldsymbol{y},\tau)G(\boldsymbol{x},t|\boldsymbol{y},\tau)dVd\tau$$

$$G(\boldsymbol{x},t|\boldsymbol{y},\tau) = \delta\left(t-\tau-\frac{|\boldsymbol{x}-\boldsymbol{y}|}{c_\infty}\right)/4\pi|\boldsymbol{x}-\boldsymbol{y}|$$





The Greens function can be thought of as a filter function which weights the contribution of each source element and adds them up at the correct retarded time

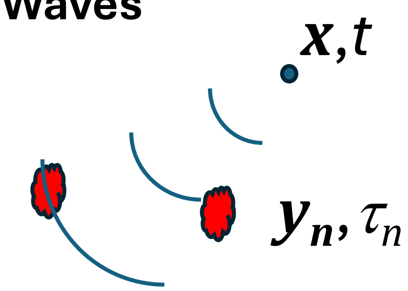
Greens Functions

$$\frac{1}{c_\infty^2}\frac{\partial^2\phi}{\partial\tau^2} - \nabla^2\phi = Q(\boldsymbol{y},\tau)$$

$$\phi(\boldsymbol{x},t) = \int_V \left[\frac{Q(\boldsymbol{y},\tau)}{4\pi|\boldsymbol{x}-\boldsymbol{y}|}\right]_{\tau=\tau^*} dV$$

retarded time $\quad \tau^* = t - \frac{|\boldsymbol{x}-\boldsymbol{y}|}{c_\infty}$

Acoustic Waves

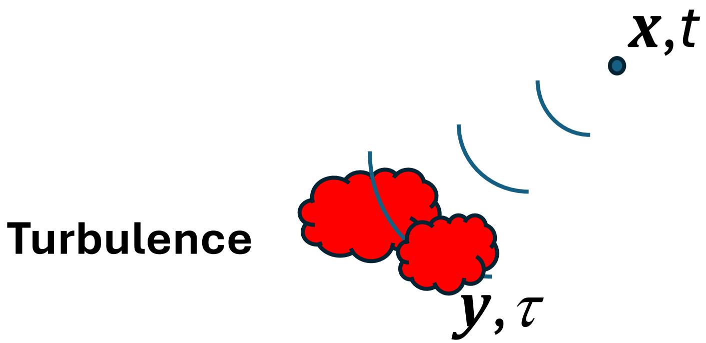


Lighthill's Equation

$$\frac{\partial^2 \rho'}{\partial t^2} - c_\infty^2 \nabla^2 \rho' = \frac{\partial^2 T_{ij}}{\partial x_i \partial x_j}$$

$$\rho' c_\infty^2(\boldsymbol{x}, t) = \int_V \frac{1}{4\pi|\boldsymbol{x}-\boldsymbol{y}|}\left[\frac{\partial^2 T_{ij}(\boldsymbol{y},\tau)}{\partial y_i \partial y_j}\right]_{\tau=\tau^*} dV$$

$$\tau^* = t - \frac{|\boldsymbol{x}-\boldsymbol{y}|}{c_\infty}$$

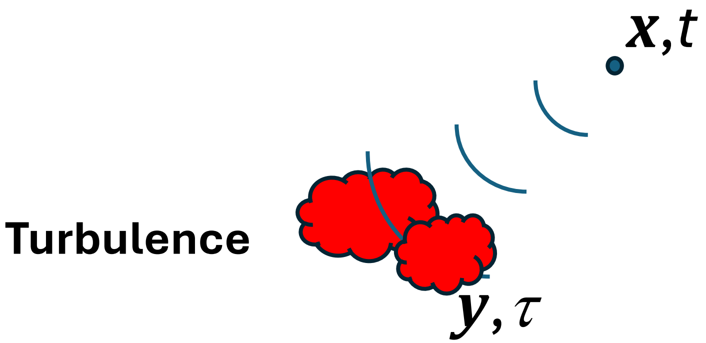


The sound in the stationary medium is the volume integral of the righthand side of Lighthill's wave equation

Lighthill's Equation for far field and small source regions $|\boldsymbol{x}|>>|\mathbf{y}|$, $V<<\lambda^3$

$$\rho' c_\infty^2(\boldsymbol{x},t) = \frac{1}{4\pi|\boldsymbol{x}|}\left[\int_V \frac{\partial^2 T_{ij}(\boldsymbol{y},\tau)}{\partial y_i \partial y_j} dV\right]_{\tau=\tau^*}$$

$$\tau^* \approx t - \frac{|\boldsymbol{x}|}{c_\infty}$$



This is a volume integral of a divergence so

$$\rho' c_\infty^2(\boldsymbol{x},t) = \frac{1}{4\pi|\boldsymbol{x}|}\left[\int_{S_\infty} \frac{\partial T_{ij}(\boldsymbol{y},\tau)}{\partial y_i} n_j dS\right]_{\tau=\tau^*} = 0$$

Use Ffowcs-Williams Observation

$$\boxed{\frac{\partial^2 \rho'}{\partial t^2} - c_\infty^2 \nabla^2 \rho' = \frac{\partial^2 T_{ij}}{\partial x_i \partial x_j}}$$

Use a modified variable

$$\frac{1}{c_\infty^2}\frac{\partial^2 \phi_{ij}}{\partial \tau^2} - \nabla^2 \phi_{ij} = T_{ij}(\boldsymbol{y}, \tau)$$

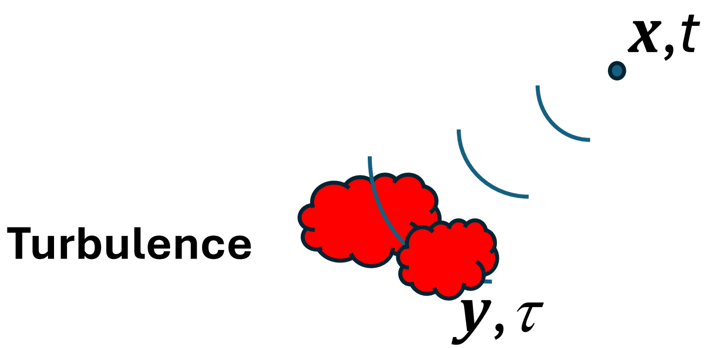
$$\phi_{ij}(\boldsymbol{x}, t) = \int_V \int_{-T}^{T} T_{ij}(\boldsymbol{y}, \tau) G(\boldsymbol{x}, t|\boldsymbol{y}, \tau) dV d\tau$$

$$\rho' c_\infty^2 = \frac{\partial^2 \phi_{ij}}{\partial x_i \partial x_j} = \frac{\partial^2}{\partial x_i \partial x_j} \int_V \int_{-T}^{T} T_{ij}(\boldsymbol{y}, \tau) G(\boldsymbol{x}, t|\boldsymbol{y}, \tau) dV d\tau$$

Lighthill's Equation

$$\rho' c_\infty^2(\boldsymbol{x},t) = \frac{\partial^2}{\partial x_i \partial x_j}\int_V \frac{1}{4\pi|\boldsymbol{x}-\boldsymbol{y}|}\left[T_{ij}(\boldsymbol{y},\tau)\right]_{\tau=\tau^*} dV$$

$$\tau^* = t - \frac{|\boldsymbol{x}-\boldsymbol{y}|}{c_\infty}$$



The sound field is now the double divergence of the propagating waves, not the source term

Lighthill's Equation in Far Field

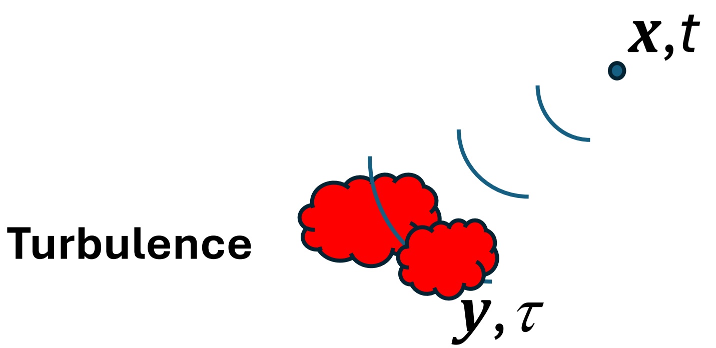
$$\rho' c_\infty^2(\boldsymbol{x},t) = \frac{\partial^2}{\partial x_i \partial x_j}\int_V \frac{1}{4\pi|\boldsymbol{x}-\boldsymbol{y}|}\left[T_{ij}(\boldsymbol{y},\tau)\right]_{\tau=\tau^*} dV$$

$$\rho' c_\infty^2(\boldsymbol{x},t) \approx \frac{1}{4\pi|\boldsymbol{x}|}\frac{\partial^2}{\partial x_i \partial x_j}\int_V \left[T_{ij}(\boldsymbol{y},\tau)\right]_{\tau=\tau^*} dV$$

$$\tau^* = t - \frac{|\boldsymbol{x}-\boldsymbol{y}|}{c_\infty}$$

$$\frac{\partial^2 f(\tau^*)}{\partial x_i \partial x_j} \approx \frac{\partial \tau^*}{\partial x_i}\frac{\partial \tau^*}{\partial x_j}\frac{\partial^2 f(\tau^*)}{\partial \tau^{*2}}$$

$$\frac{\partial \tau^*}{\partial x_i} \approx \frac{1}{c_\infty}\frac{x_i}{|\boldsymbol{x}|}$$



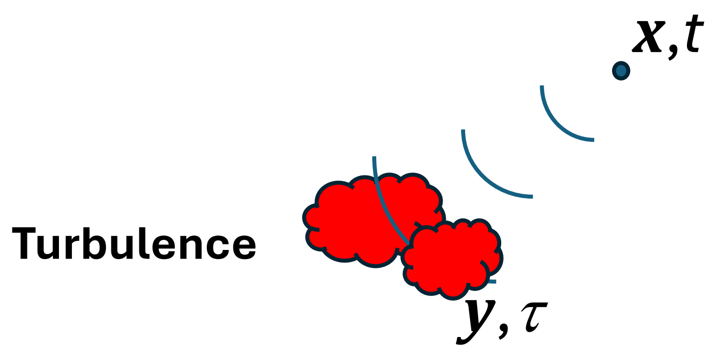
The rate of change of the wave motion is much faster than the decay with distance

Lighthill's Equation in the Far Field

$$\rho' c_\infty^2(\boldsymbol{x},t) = \frac{\partial^2}{\partial x_i \partial x_j}\int_V \frac{1}{4\pi|\boldsymbol{x}-\boldsymbol{y}|}\left[T_{ij}(\boldsymbol{y},\tau)\right]_{\tau=\tau^*} dV$$

$$\rho' c_\infty^2(\boldsymbol{x},t) \approx \frac{x_i x_j}{4\pi|\boldsymbol{x}|^3 c_\infty^2}\int_V \left[\frac{\partial^2 T_{ij}(\boldsymbol{y},\tau)}{\partial \tau^2}\right]_{\tau=\tau^*} dV$$

$$\tau^* = t - \frac{|\boldsymbol{x}-\boldsymbol{y}|}{c_\infty}$$



Note the quadrupole type directionality and the speed of sound dependence

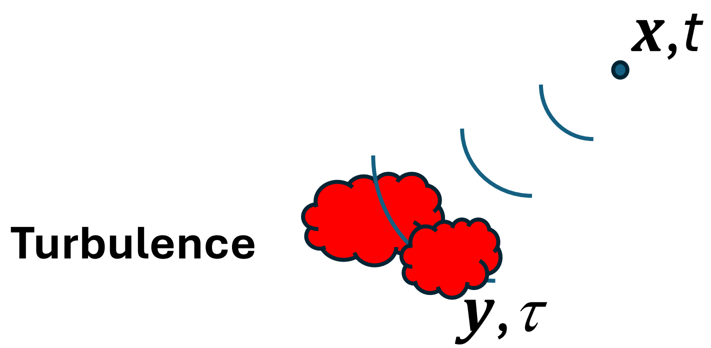
Scaling on Flow Speed

$$\rho' c_\infty^2(\boldsymbol{x},t) \approx \frac{x_i x_j}{4\pi|\boldsymbol{x}|^3 c_\infty^2}\int_V \left[\frac{\partial^2 T_{ij}(\boldsymbol{y},\tau)}{\partial\tau^2}\right]_{\tau=\tau^*} dV$$

$$T_{ij} = \rho v_i v_j \sim \rho_o U^2$$

$$\tau \sim L/U$$

$$\rho' c_\infty^2(\boldsymbol{x},t) = p \sim \frac{\rho_o x_i x_j U^4 V}{4\pi|\boldsymbol{x}|^3 c_\infty^2 L^2}$$

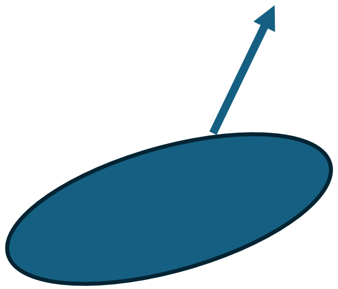


$$p^2 \sim \left(\frac{\rho_o x_i x_j V}{4\pi|\boldsymbol{x}|^3 c_\infty^2 L^2}\right)^2 U^8$$

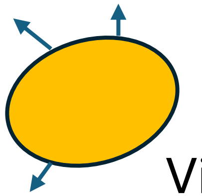
- **The mean square pressure, or intensity, scales with the Eighth power of the flow speed**
- **Halving the flow speed gives a 24dB noise reduction**

Adding Surfaces to the Flow

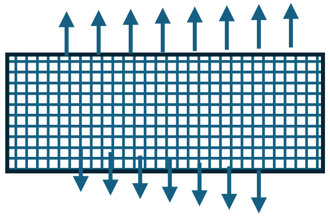
The Ffowcs Williams and Hawkings Equation



Solid body in motion

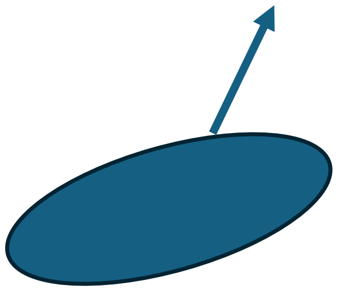


Vibrating surface

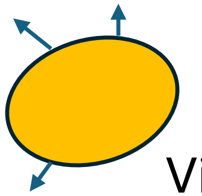


Permeable surface
such as a CFD grid

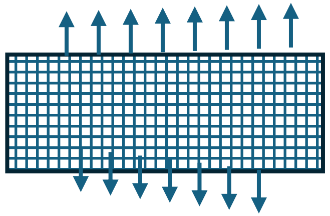
The Ffowcs Williams and Hawkings Equation



Solid body in motion

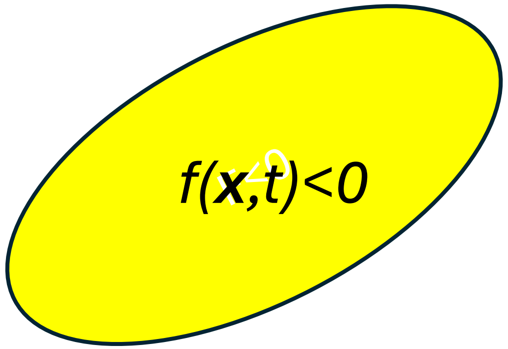


Vibrating surface



Permeable surface
such as a CFD grid

$f(\mathbf{x},t)>0$

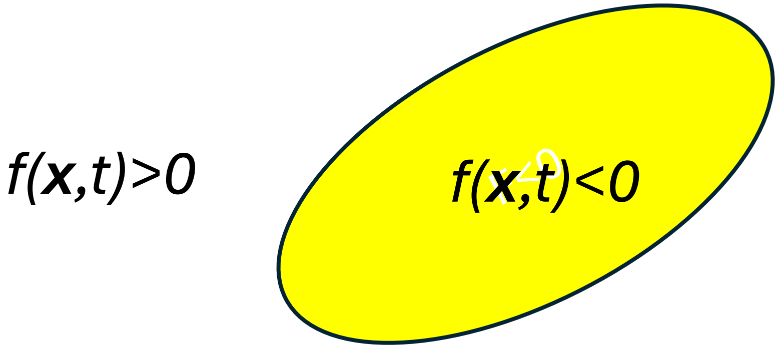


The surface is defined as a function of space and time by the function $f(\mathbf{x},t)=0$

$f(\mathbf{x},t)=x_1^2+x_2^2+(x_3-Ut)^2-a^2$

sphere moving at constant speed

The Ffowcs Williams and Hawkings Equation



Define new variables

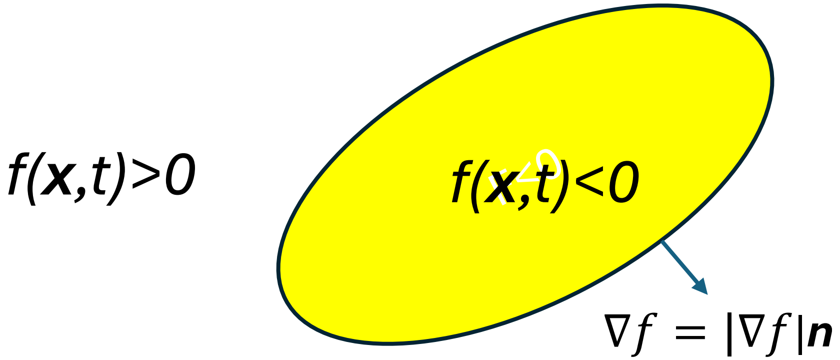
$\boldsymbol{v}H_s(f)$ $\quad$ $pH_s(f)$ $\quad$ $\rho H_s(f)$

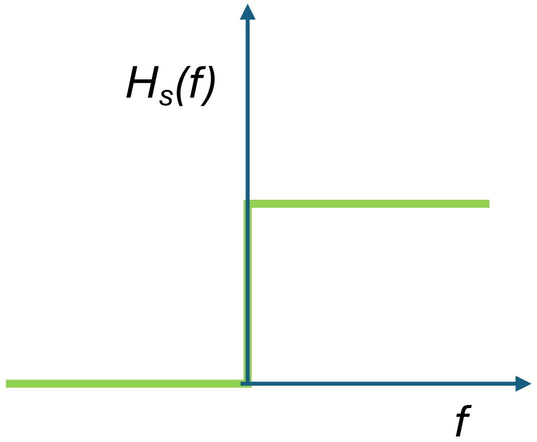
$H_s(f) = 0.$ *inside surface*

$H_s(f) = 1.$ *outside surface*

Each new variable is zero inside the surface but equal to it's defined value outside the surface

The Ffowcs Williams and Hawkings Equation





Operators applied to new variables

$$\boldsymbol{\nabla}.\left(\boldsymbol{v}H_s(f)\right) = H_s(f)\boldsymbol{\nabla}.(\boldsymbol{v}) + \boldsymbol{v}.\boldsymbol{\nabla}H_s(f)$$

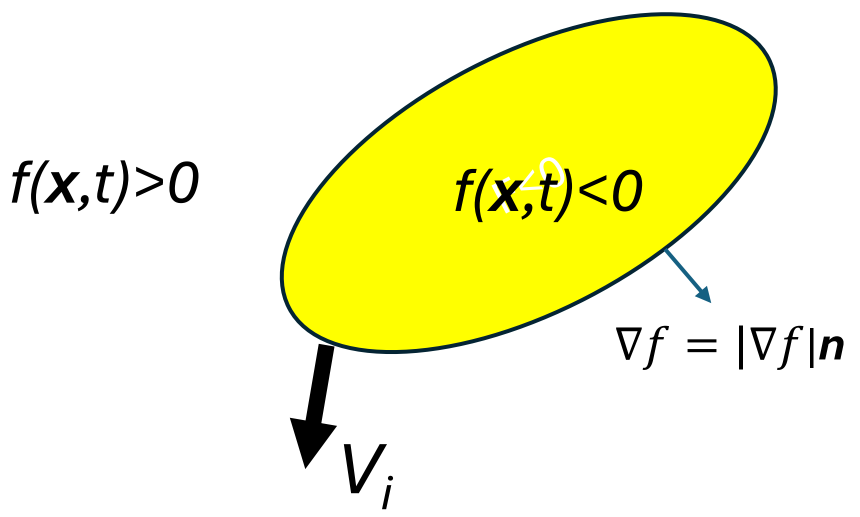
$$\boldsymbol{\nabla}H_s(f) = \frac{\partial H_s(f)}{\partial f}\nabla f = \delta(f)\,\nabla f$$

$$\boldsymbol{\nabla}.\left(\boldsymbol{v}H_s(f)\right) = H_s(f)\boldsymbol{\nabla}.(\boldsymbol{v}) + \boldsymbol{v}.\boldsymbol{n}|\nabla f|\delta\,(f)$$

Divergence of variable outside surface
but is zero inside the surface

Velocity normal to surface

The Ffowcs Williams and Hawkings Equation



New Continuity equation

$$\frac{\partial(\rho' H_s)}{\partial t} + \boldsymbol{\nabla}.(\rho \boldsymbol{v} H_s) = H_s \cancelto{\text{zero}}{\left(\frac{\partial \rho'}{\partial t} + \boldsymbol{\nabla}.(\rho \boldsymbol{v})\right)} + (\rho \boldsymbol{v} - \rho' \boldsymbol{V}).\boldsymbol{n}|\nabla f|\delta\,(f)$$

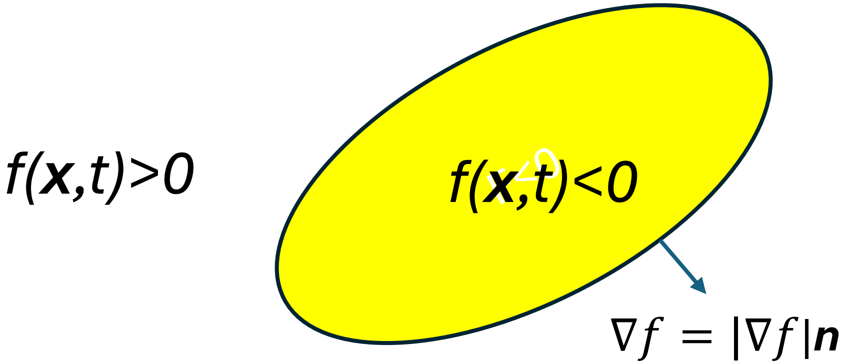
$$\frac{\partial(\rho' H_s)}{\partial t} + \frac{\partial}{\partial x_i}(\rho v_i H_s) = (\rho v_i - \rho' V_i).n_i|\nabla f|\delta\,(f)$$

Continuity equation

$$\frac{\partial(\rho v_i H_s)}{\partial t} + \frac{\partial}{\partial x_j}\left(\rho v_i v_j H_s + p_{ij} H_s\right) = \left(\rho v_i (v_j - V_j) + p_{ij}\right) n_j |\nabla f|\delta\,(f)$$

Momentum equation

The Ffowcs Williams and Hawkings Equation



New Wave Equation

$$\frac{\partial^2(\rho' H_s)}{\partial t^2} - c_\infty^2 \frac{\partial^2(\rho' H_s)}{\partial x_i^2}$$

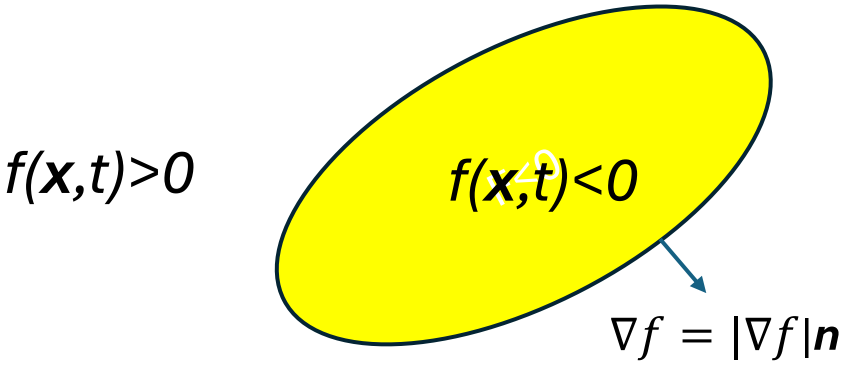
$$= \frac{\partial^2\left(T_{ij} H_s\right)}{\partial x_i \partial x_j} - \frac{\partial}{\partial x_i}\left(\left(\rho v_i\left(v_j - V_j\right) + p_{ij}\right) n_j |\nabla f| \delta\left(f\right)\right) + \frac{\partial}{\partial t}\left(\left(\rho v_i - \rho' V_i\right).n_i |\nabla f| \delta\left(f\right)\right)$$

Surface momentum flux and pressure stress

Surface Motion Term

Lighthill's Source

The solution to the FW-H Equation



$$G(\boldsymbol{x}, t|\boldsymbol{y}, \tau) = \delta\left(t - \tau - \frac{|\boldsymbol{x} - \boldsymbol{y}|}{c_\infty}\right)/4\pi|\boldsymbol{x} - \boldsymbol{y}|$$

$$\rho' c_\infty^2 H_s$$

Lighthill's Source

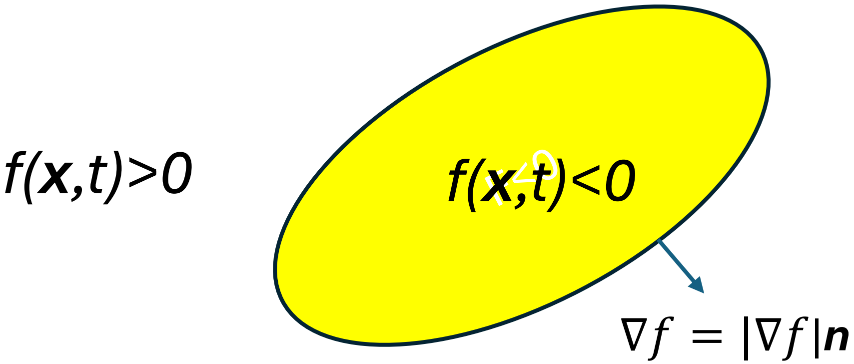
$$= \frac{\partial^2}{\partial x_i \partial x_j} \int_V \int_{-T}^{T} T_{ij} H_s G dV d\tau$$

Surface momentum flux and pressure stress

$$- \frac{\partial}{\partial x_i} \int_V \int_{-T}^{T} \left(\left(\rho v_i (v_j - V_j)\right) + p_{ij}\right) n_j |\nabla f| \delta\,(f)\right) G dV d\tau$$

$$+ \frac{\partial}{\partial t} \int_V \int_{-T}^{T} \left(\left(\rho v_i - \rho' V_i\right). n_i |\nabla f| \delta\,(f)\right) G dV d\tau$$ **Surface Motion Term**

The solution to the FW-H Equation



$$G(\boldsymbol{x},t|\boldsymbol{y},\tau) = \frac{\delta\left(t-\tau-\frac{r(\tau)}{c_\infty}\right)}{4\pi r(\tau)}$$

$$r(\tau) = |\boldsymbol{x}-\boldsymbol{y}(\tau)|$$

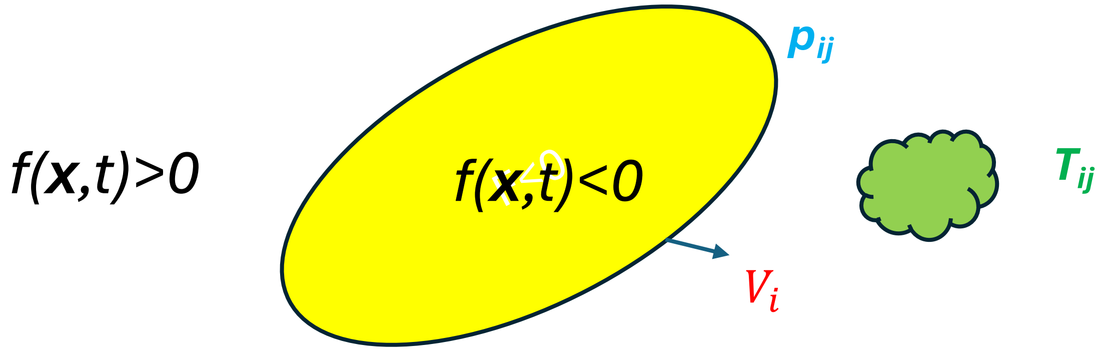
$$\int_V \int_{-T}^{T} q_j(\boldsymbol{y},\tau)n_j|\nabla f|\delta\,(f))G(\boldsymbol{x},t|\boldsymbol{y},\tau)dVd\tau$$

$$\frac{1}{c_\infty}\frac{\partial r}{\partial \tau} = -M_r$$

$$= \int_S \int_{-T}^{T} q_j(\boldsymbol{y},\tau)n_j G(\boldsymbol{x},t|\boldsymbol{y},\tau)dSd\tau$$

$$= \int_S \left[\frac{q_j(\boldsymbol{y},\tau)n_j}{4\pi r(\tau)}\right]_{\tau=\tau^*} dS$$

The solution to the FW-H Equation



$$\rho' c_\infty^2 H_s = \frac{\partial^2}{\partial x_i \partial x_j} \int_V \left[\frac{T_{ij} H_s}{4\pi r(\tau)|1 - M_r|}\right]_{\tau=\tau^*} dV$$

Lighthill's Source

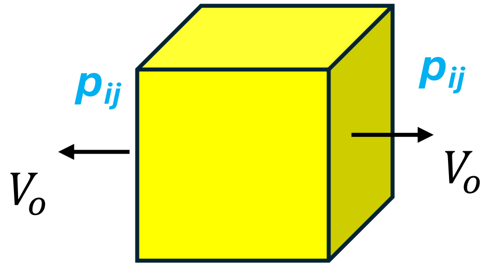
$$- \frac{\partial}{\partial x_i} \int_S \left[\frac{\left(\rho v_i (v_j - V_j) + p_{ij}\right) n_j}{4\pi r(\tau)|1 - M_r|}\right]_{\tau=\tau^*} dS$$

Surface momentum flux and pressure stress

$$+ \frac{\partial}{\partial t} \int_S \left[\frac{(\rho v_i - \rho' V_i).n_i}{4\pi r(\tau)|1 - M_r|}\right]_{\tau=\tau^*} dS$$

Surface Motion Term

Some Examples- Cube with 2 sides moving



The surface is impenetrable so ***v*=*V***

$$(\rho v_i - \rho' V_i)n_i = (\rho v_i - (\rho - \rho_\infty)V_i)n_i = \rho_\infty V_i n_i$$

$$\rho' c_\infty^2 H_s = \frac{\partial^2}{\partial x_i \partial x_j} \int_V \left[\frac{T_{ij} H_s}{4\pi r(\tau)|1 - M_r|}\right]_{\tau=\tau^*} dV$$

Lighthill's Source

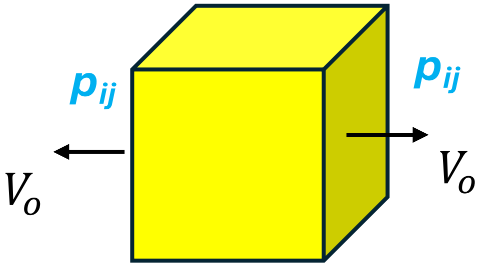
$$-\frac{\partial}{\partial x_i} \int_S \left[\frac{(\cancel{\rho v_i (v_j - V_j)} + p_{ij})n_j}{4\pi r(\tau)|1 - M_r|}\right]_{\tau=\tau^*} dS$$

Surface momentum flux and pressure stress

$$+\frac{\partial}{\partial t} \int_S \left[\frac{(\rho v_i - \rho' V_i).n_i}{4\pi r(\tau)|1 - M_r|}\right]_{\tau=\tau^*} dS$$

Surface Motion Term

Some Examples- Cube with 2 sides moving



- Assume observer is in the far field so $r=|\mathbf{x}|$
- Ignore the non linear terms
- Assume $V_o << c_\infty$

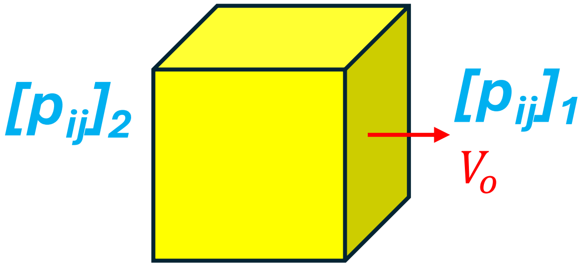
$$\rho' c_\infty^2 H_s = \cancel{\frac{\partial^2}{\partial x_i \partial x_j} \int_V \left[\frac{T_{ij} H_s}{4\pi r(\tau)|1 - M_r|} \right]_{\tau=\tau^*} dV}$$

$$- \frac{1}{4\pi|\boldsymbol{x}|} \frac{\partial}{\partial x_i} \int_S [p_{ij} n_j]_{\tau=\tau^*} dS$$ Net Surface Force

The net force is zero so only the volume displacement term remains

$$+ \frac{1}{4\pi|\boldsymbol{x}|} \frac{\partial}{\partial t} \int_S [\rho_\infty V_i . n_i]_{\tau=\tau^*} dS$$ **Surface Motion Term**

Some Examples- Cube with One side moving



- Assume observer is in the far field so $r=|\mathbf{x}|$
- Ignore the non linear terms
- Assume $V_o << c_\infty$

$$\rho' c_\infty^2 H_s =$$

Net Surface Force

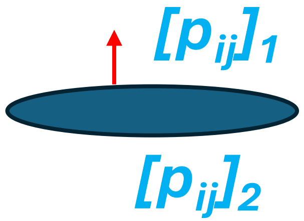
$$-\frac{1}{4\pi|\boldsymbol{x}|}\frac{\partial}{\partial x_i}\int_S [p_{ij}n_j]_{\tau=\tau^*}dS$$

The net force is <u>not</u> zero so both terms are needed

$$+\frac{1}{4\pi|\boldsymbol{x}|}\frac{\partial}{\partial t}\int_S [\rho_\infty V_i . n_i]_{\tau=\tau^*}dS$$

Surface Motion Term

Some Examples- Stationary Airfoil with unsteady load



- Assume observer is in the far field so $r=|\mathbf{x}|$
- Ignore the non linear terms
- Assume $V_o=0$

$$\rho' c_\infty^2 H_s = \cancel{\frac{\partial^2}{\partial x_i \partial x_j}\int_V \left[\frac{T_{ij}H_s}{4\pi r(\tau)|1-M_r|}\right]_{\tau=\tau^*} dV}$$

The net force is the only term remaining

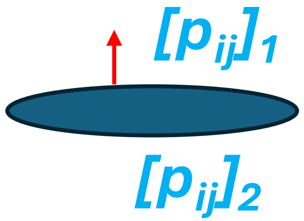
$$-\frac{1}{4\pi|\boldsymbol{x}|}\frac{\partial}{\partial x_i}\int_S \left[p_{ij}n_j\right]_{\tau=\tau^*} dS$$

Net Surface Force

$$+\cancel{\frac{1}{4\pi|\boldsymbol{x}|}\frac{\partial}{\partial t}\int_S \left[\rho_\infty V_i . n_i\right]_{\tau=\tau^*} dS}$$

Surface Motion Term

Some Examples- Stationary Airfoil with unsteady load



- Assume observer is in the far field so $r=|\mathbf{x}|$
- Ignore the non linear terms
- Assume $V_o=0$

Pressure integrated at the correct retarded time

$$-\frac{1}{4\pi|\boldsymbol{x}|}\frac{\partial}{\partial x_i}\int_S \left[p_{ij}n_j\right]_{\tau=\tau^*} dS = \frac{x_i}{4\pi|\boldsymbol{x}|^2 c_\infty}\frac{\partial}{\partial t}\int_S \left[p_{ij}n_j\right]_{\tau=\tau^*} dS$$

$$= \frac{\boldsymbol{x_i}}{\boldsymbol{4\pi|x|^2 c_\infty}}\left[\frac{\boldsymbol{\partial F_i(\tau)}}{\boldsymbol{\partial\tau}}\right]_{\boldsymbol{\tau=\tau^*}}$$

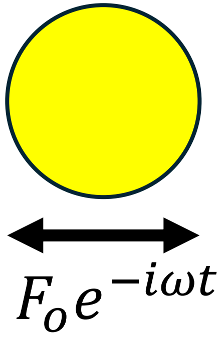
Net Surface Force if the surface is compact

$$p \sim \rho_\infty U^3 A / L c_\infty r$$

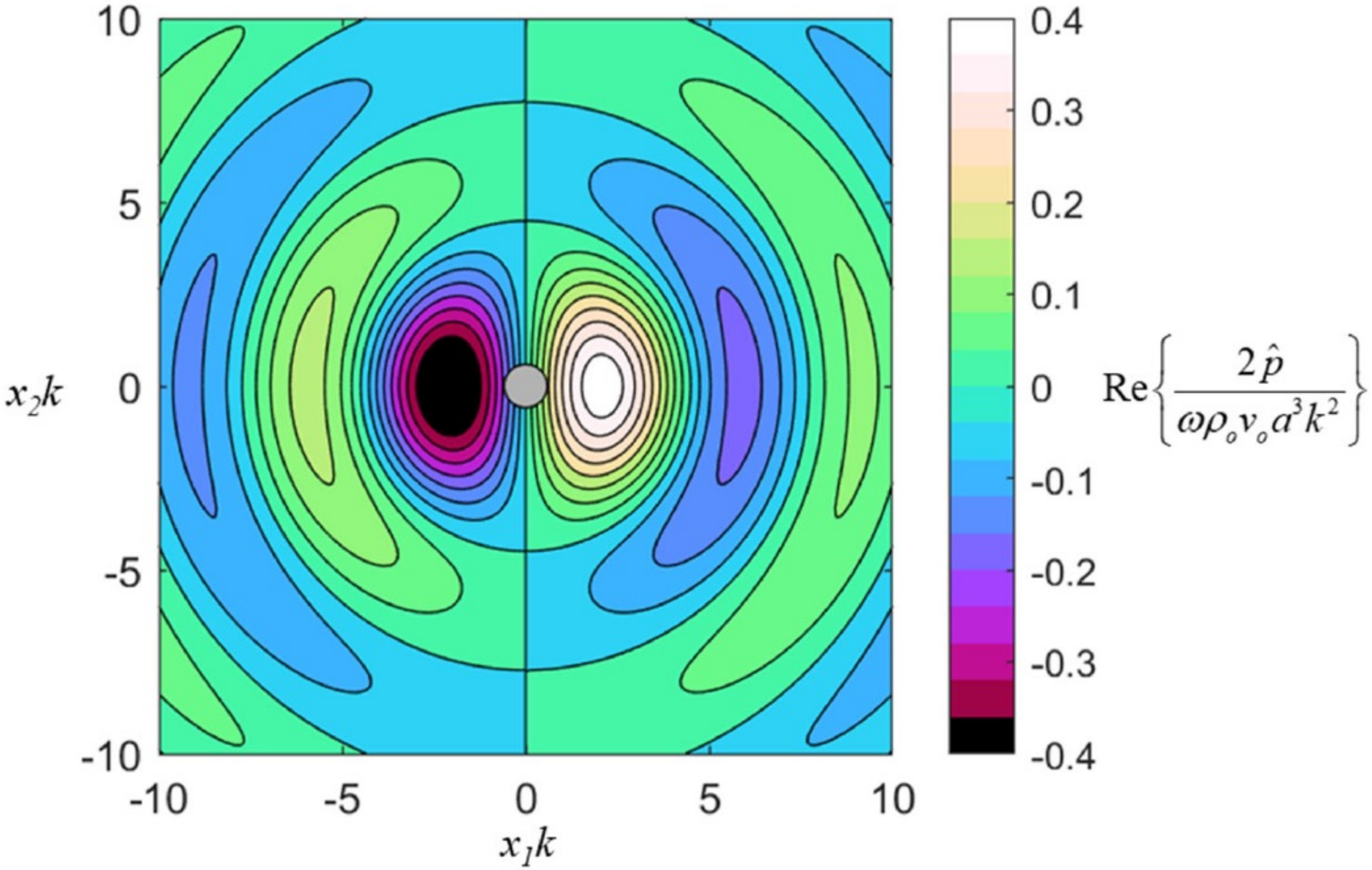
$F_i \sim \rho_\infty U^2 A$ $\quad$ $t \sim L/U.$

$$p^2 \sim U^6$$

Cosine Directionality



$$\frac{x_1}{r} = cos\theta$$



$$\hat{p}(r) = -ik\cos\theta\, e^{ikr}\left(\frac{\hat{F}}{4\pi r}\right)\left(1 - \frac{1}{ikr}\right)$$

Loading Noise Scales as U^6

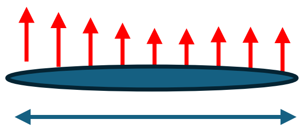
$$p^2 \sim U^6$$

- Halving the flow speed gives an 18 dB noise reduction for no change in shape

- Since lift $\sim U^2 A$ and noise as $\sim U^6 A^2 = U^2 (U^2 A)^2$ halving speed at constant lift gives a 6dB noise reduction

- For a propeller thrust $\sim U_{tip}^2 R^2$ and noise $\sim U_{tip}^6 R^4 = (U_{tip}^2 R^2)^3 / R^2$ so noise is reduced by increasing the diameter

Some Examples- Large Flat Surface



- Assume observer is in the far field so $r=|\mathbf{x}|>>c>>\lambda$
- Ignore the non linear terms
- Assume $V_o=0$

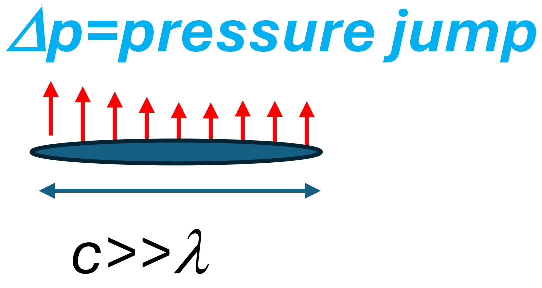
$$-\frac{1}{4\pi|\boldsymbol{x}|}\frac{\partial}{\partial x_i}\int_S \left[p_{ij}n_j\right]_{\tau=\tau^*}dS=\frac{x_1}{4\pi|\boldsymbol{x}|^2c_\infty}\frac{\partial}{\partial t}\int_\Sigma \left[\Delta p\right]_{\tau=\tau^*}d\Sigma$$

$$\Delta p(\boldsymbol{y},\tau)=P_o(y_1,y_3)\exp(-i\omega t)$$

Far Field Approximation for the correct retarded time

$$\boldsymbol{\frac{-i\omega x_1}{4\pi|x|^2c_\infty}\int_0^c\int_{-\frac{b}{2}}^{\frac{b}{2}} P_o(y_1,y_3)e^{-i\omega t-ik|x|+ikx.y/|x|}dy_1dy_3}$$

Some Examples- Large Flat Surface



- Assume observer is in the far field so $r=|\mathbf{x}|>>c>>\lambda$
- Ignore the non linear terms
- Assume $V_o=0$

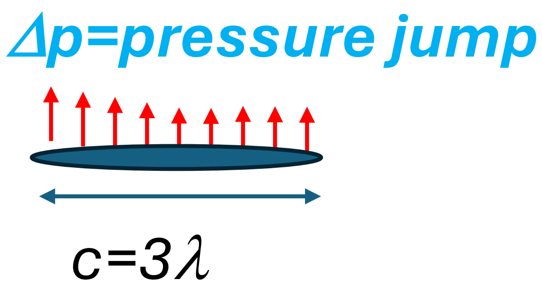
$$P_o(y_1, y_3) = P_o = const$$

$$\frac{-i\omega x_1 P_o e^{-i\omega t - ik|x|}}{4\pi|x|^2 c_\infty}\int_0^c \int_{-\frac{b}{2}}^{\frac{b}{2}} e^{ik(x_1.y_1+x_3y_3)/|x|}dy_1dy_3$$

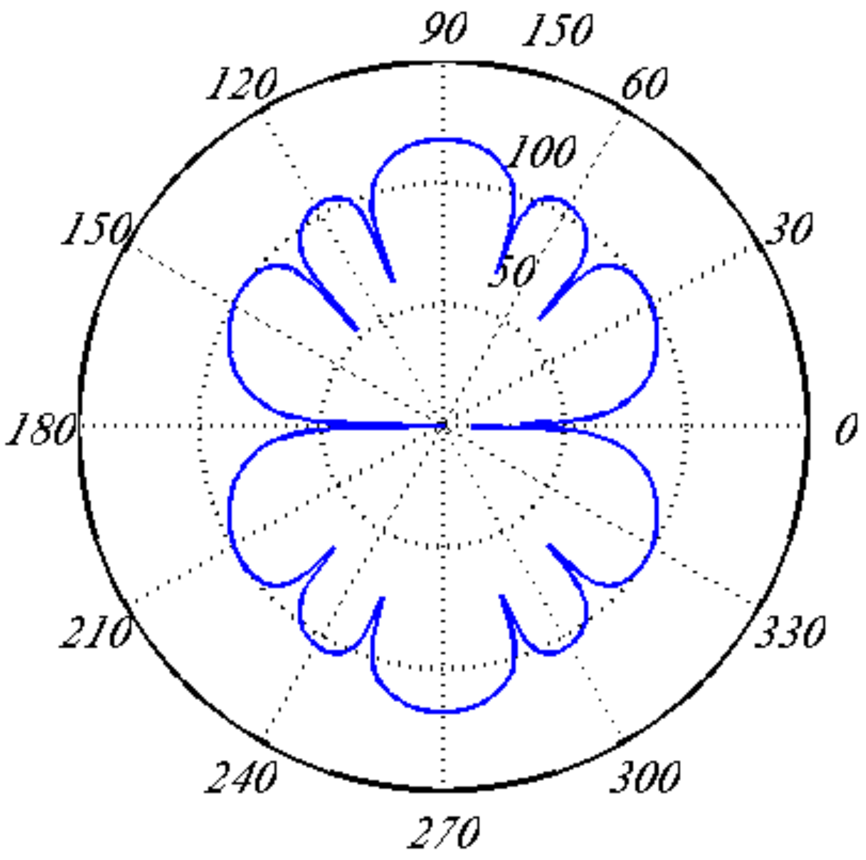
$$=\frac{-i\omega x_1 bcP_o e^{-i\omega t - ik|x|}}{4\pi|x|^2 c_\infty}\left\{\frac{sin\left(\frac{kx_1c}{2|x|}\right)}{\frac{kx_1c}{2|x|}}\right\}\left\{\frac{sin\left(\frac{kx_3b}{2|x|}\right)}{\frac{kx_3b}{2|x|}}\right\}$$

Additional Directionality Terms

Some Examples- Large Flat Surface



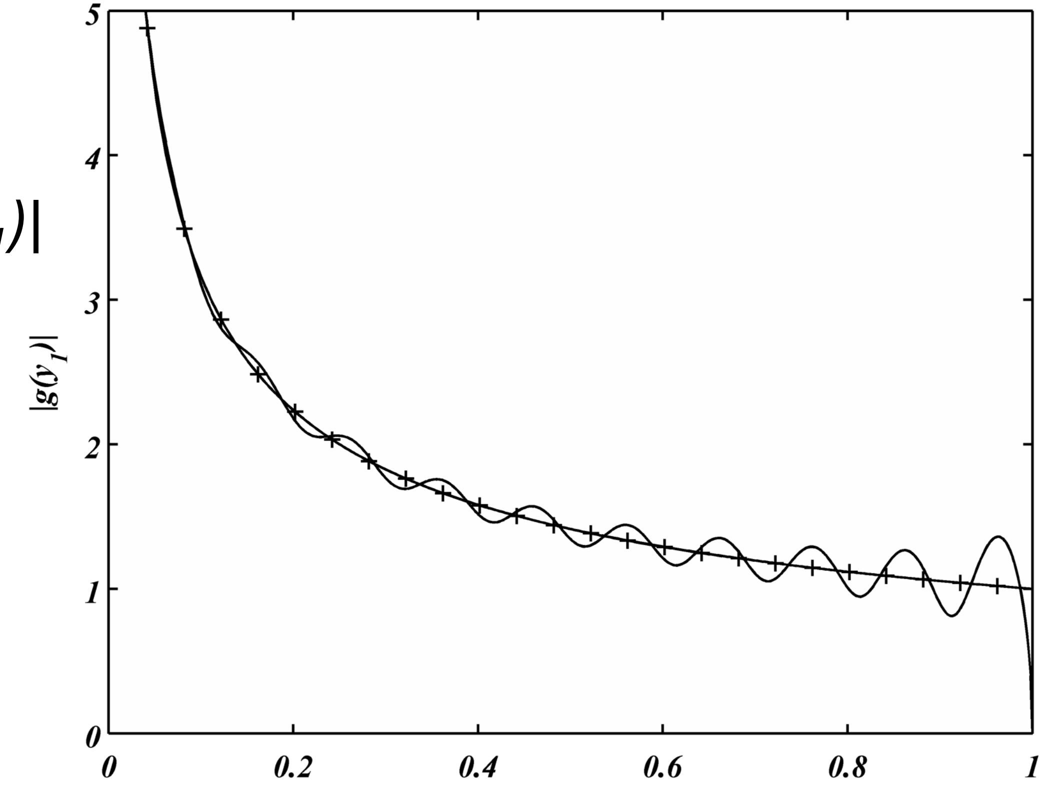
$$P_o(y_1, y_3) = P_o = const$$



$$\frac{-i\omega x_1 b c P_o e^{-i\omega t - ik|x|}}{4\pi |x|^2 c_\infty} \left\{ \frac{\sin\left(\frac{k x_1 c}{2|x|}\right)}{\frac{k x_1 c}{2|x|}} \right\} \left\{ \frac{\sin\left(\frac{k x_3 b}{2|x|}\right)}{\frac{k x_3 b}{2|x|}} \right\}$$

Unsteady Loading Distribution on Airfoil

$|\Delta p(y_1)|$

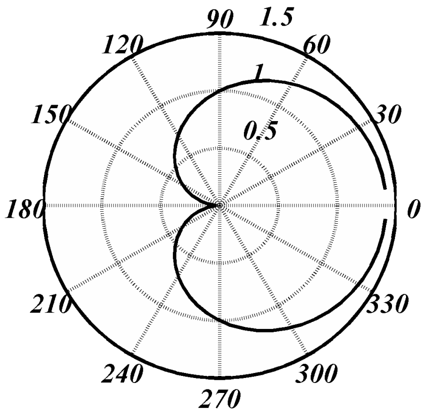


y_1/c

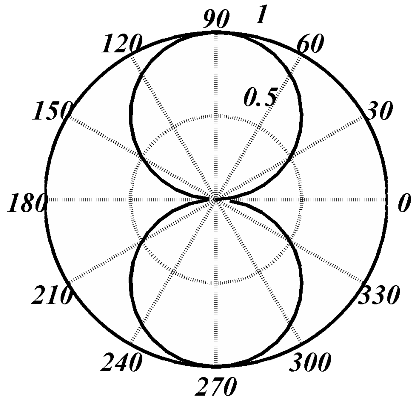
Far Field Directionality

Semi-Infinite
Blade Chord

$(a) \quad k_o c = \infty$



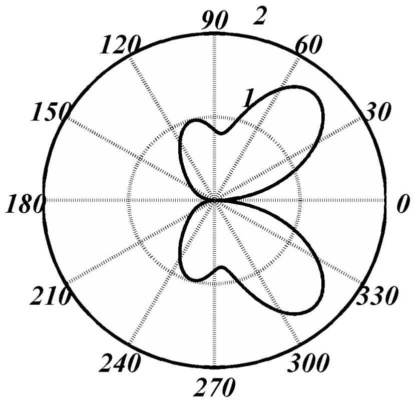
$(b) \quad k_o c << 1$



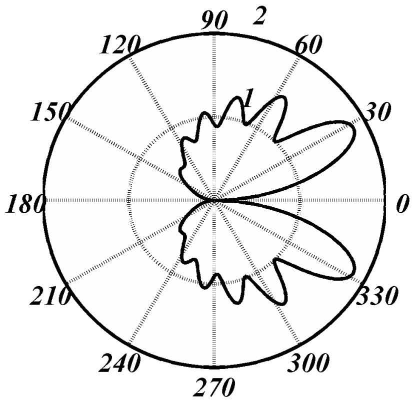
Chord<<λ

Chord=λ

$(c) \quad k_o c = 2\pi$



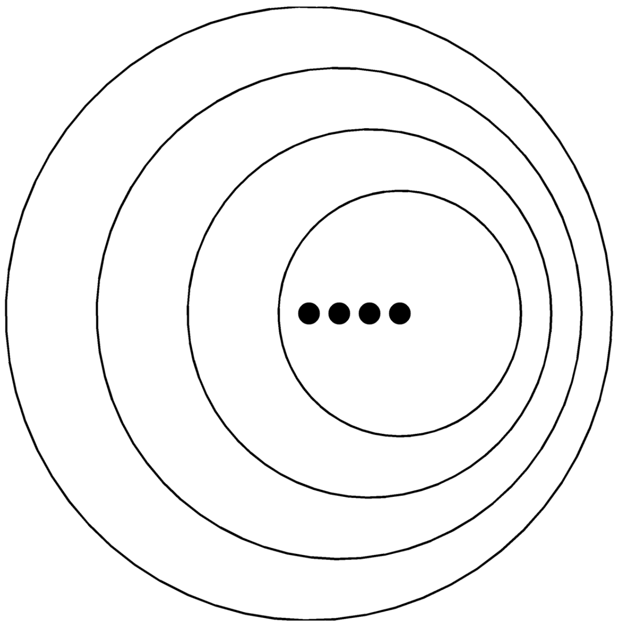
$(d) \quad k_o c = 6\pi$



Chord=3λ

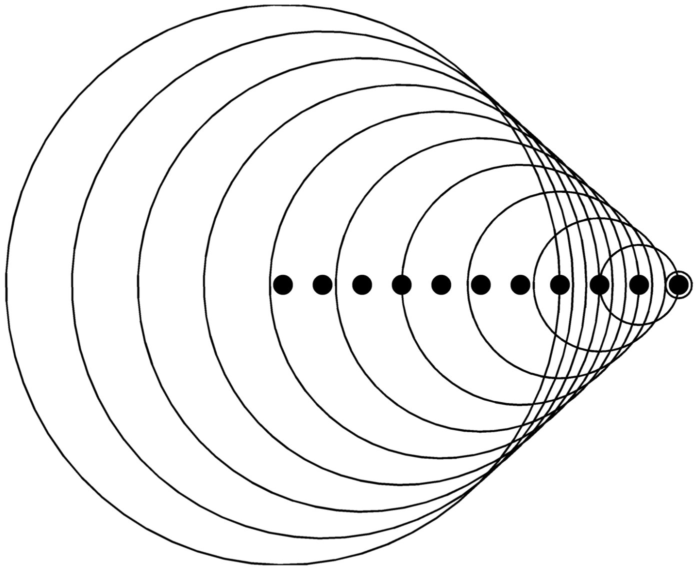
Moving Sources

Moving Sources



Subsonic Motion

(a)



Supersonic Motion

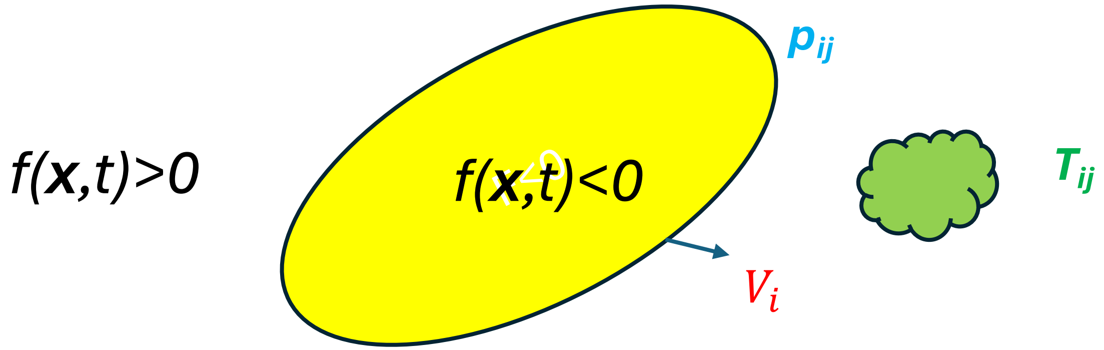
(b)

$$t = \tau + |\boldsymbol{x} - \boldsymbol{y}(\tau)|/c_\infty$$

$$\frac{\partial f(\tau)}{\partial t} = \frac{\partial \tau}{\partial t}\frac{\partial f}{\partial \tau} = \frac{1}{1 - M_r}\frac{\partial f}{\partial \tau}$$

$$M_r = V cos\theta / c_\infty$$

The solution to the FW-H Equation



$$\rho' c_\infty^2 H_s = \frac{\partial^2}{\partial x_i \partial x_j} \int_V \left[\frac{T_{ij} H_s}{4\pi r(\tau)|1 - M_r|}\right]_{\tau=\tau^*} dV$$

Lighthill's Source

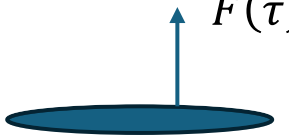
$$- \frac{\partial}{\partial x_i} \int_S \left[\frac{p_{ij} n_j}{4\pi r(\tau)|1 - M_r|}\right]_{\tau=\tau^*} dS$$

Surface momentum flux and pressure stress

$$+ \frac{\partial}{\partial t} \int_S \left[\frac{\rho_\infty V_i . n_i}{4\pi r(\tau)|1 - M_r|}\right]_{\tau=\tau^*} dS$$

Surface Motion Term

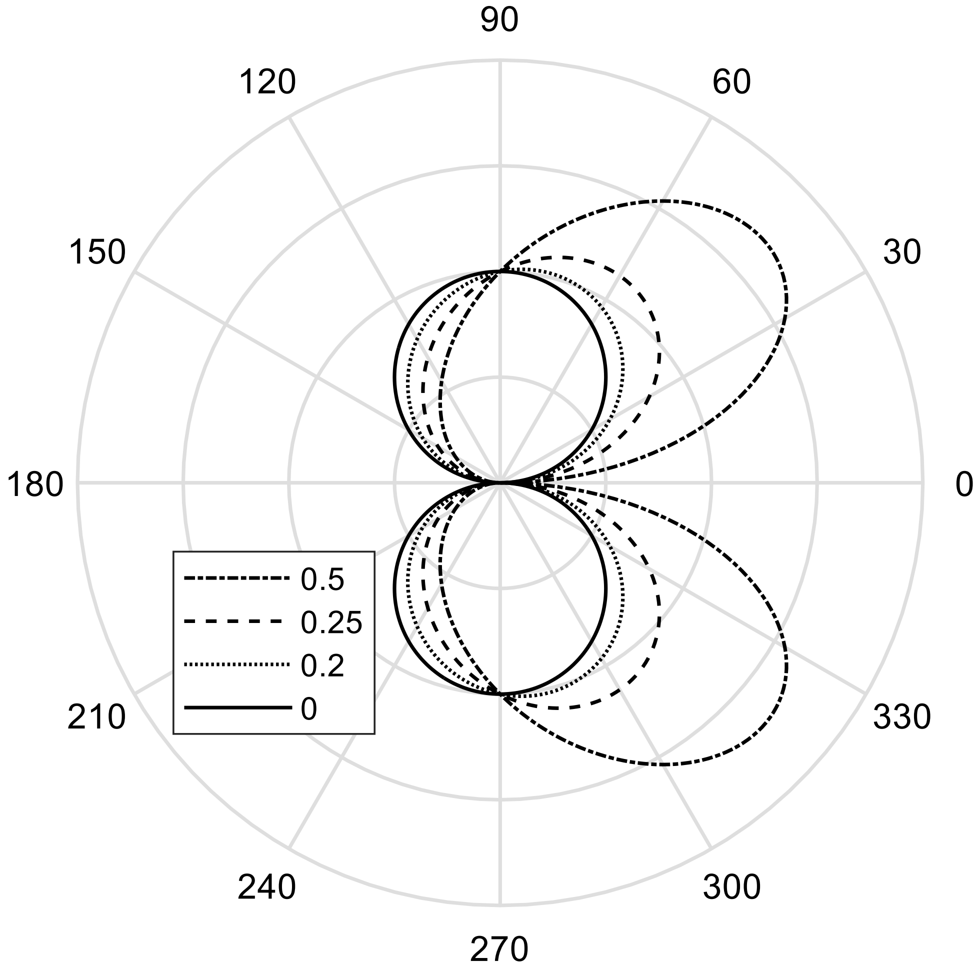
Dipole in Motion



$F(\tau) = F_o e^{-i\omega_o \tau}$

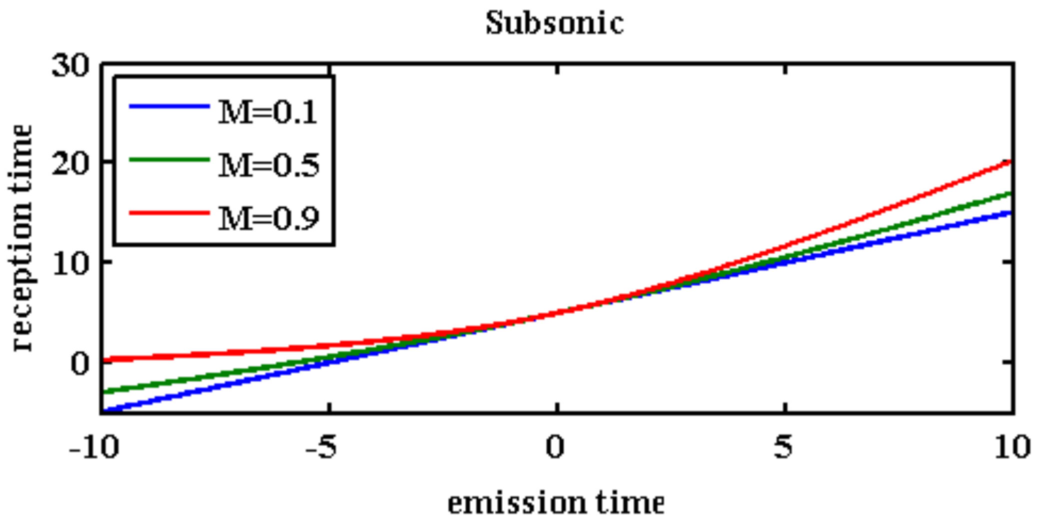
In the far field for a harmonic source

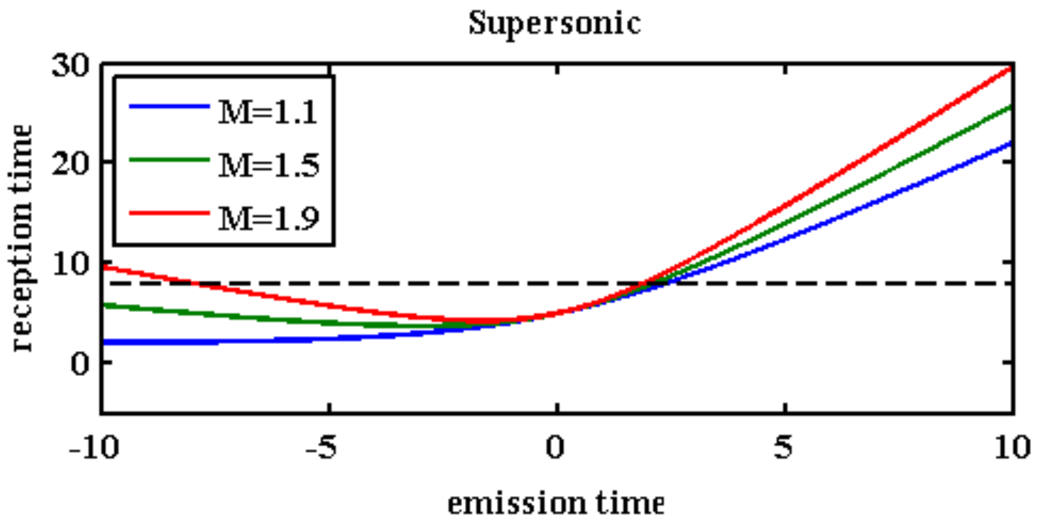
$$\rho' c_\infty^2 H_s = \left[\frac{-i\omega_o x_2 F_o e^{-i\omega_o \left(t - \frac{|\boldsymbol{x}|}{c_\infty} \right)/(1-M_r)}}{4\pi |\boldsymbol{x}|^2 (1 - M_r)^2} \right]$$

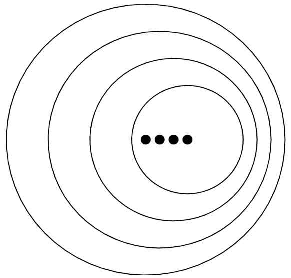


Note the Doppler amplification caused by the source motion

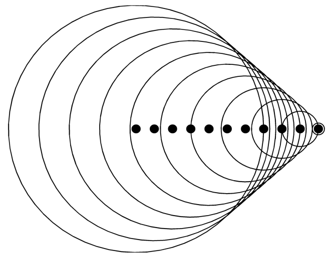
Reception and Emission Time







(a)

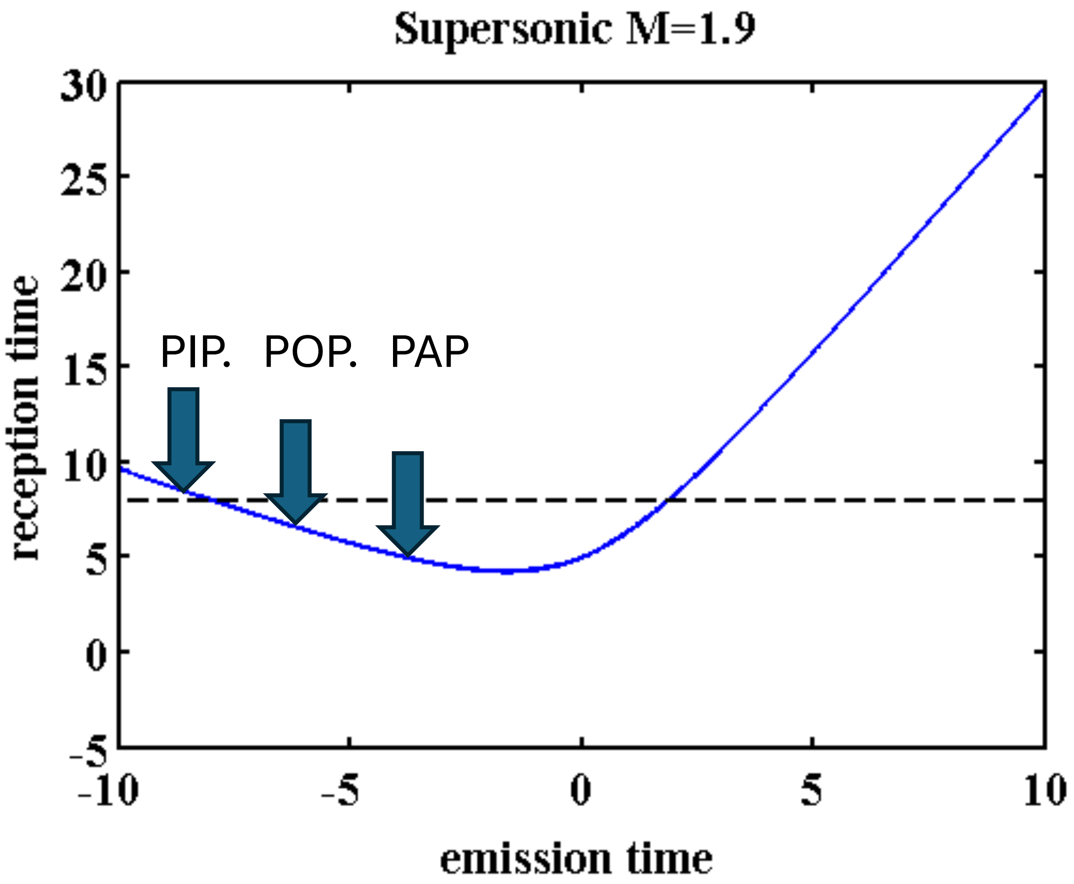


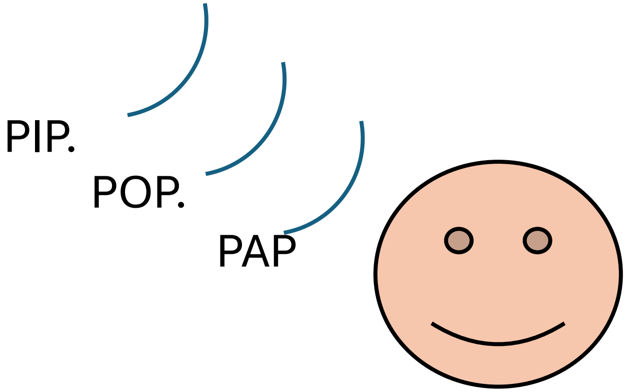
(b)

For subsonic source there is only one emission time for each reception time

For supersonic source there can be two emission times for the same reception time

PIP POP PAP. (Lighthill)





The observer hears the
sound signature in reverse